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The alternating marked point process of h-slopes of drifted Brownian motion

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Abstract

We show that the slopes between h-extrema of the drifted 1D Brownian motion form a stationary alternating marked point process, extending the result of J. Neveu and J. Pitman for the non-drifted case. Our analysis covers the results on the statistics of h-extrema obtained by P. Le Doussal, C. Monthus and D. Fisher via a Renormalization Group analysis and gives a complete description of the slope between h-extrema covering the origin by means of the Palm–Khinchin theory. Moreover, we analyze the behavior of the Brownian motion near its h-extrema.

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Keywords: Brownian motion; Marked point processes; Palm–Khinchin theory; Fluctuation theory

1. Introduction

Let B be a two-sided standard Brownian motion with drift $-\mu$. Given $h > 0$ we say that B admits an *h-minimum* at $x \in \mathbb{R}$, and that x is a point of h-minimum, if there exist $u < x < v$ such that $B_t \geq B_x$ for all $t \in [u, v]$, $B_u \geq B_x + h$ and $B_v \geq B_x + h$. Similarly, we say that B admits an *h-maximum* at $x \in \mathbb{R}$, and that x is a point of h-maximum, if there exist $u < x < v$ such that $B_t \leq B_x$ for all $t \in [u, v]$, $B_u \leq B_x - h$ and $B_v \leq B_x - h$. We say that B admits an *h-extremum* at $x \in \mathbb{R}$, and that x is a point of h-extremum, if x is a point of h-minimum or a point of h-maximum. Finally, the truncated trajectory B going from an h-minimum to an h-maximum

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will be called upward h-slope, while the truncated trajectory B going from an h-maximum to an h-minimum will be called downward h-slope.

Our first object of investigation is the statistics of h-slopes. The non-drifted case $\mu = 0$ has been studied in [1]. Here we assume $\mu \neq 0$ and show (see Theorem 1) that the statistics of h-slopes is well described by a stationary alternating marked simple point process on $\mathbb{R}$ whose points are the points of h-extrema of the Brownian motion, and each point x is marked by the h-slope going from x to the subsequent point of h-extremum. We will show that the h-slopes are independent and specify the laws P_+^μ, P_-^μ of upward h-slopes and downward h-slopes not covering the origin, respectively. The h-slope covering the origin shows a different distribution that can be derived by means of the Palm–Khinchin theory [2,3].

Our proof is based both on fluctuation theory for Lévy processes, and on the theory of marked simple point processes. The part of fluctuation theory follows strictly the scheme of [1] and can be generalized to spectrally one-sided Lévy processes, i.e. real valued random processes with stationary independent increments and with no positive jumps or with no negative jumps [4] [Chapter VII]. In fact, some of the identities of Lemma 1 and Proposition 1 below have already been obtained with more sophisticated methods for general spectrally one-sided Lévy processes (see [5–7] and references therein). On the other hand, the description of the h-slopes as a stationary alternating marked simple point process allows using the very powerful Palm–Khinchin theory, which extends renewal theory and leads to a complete description of the h-slope covering the origin. This analysis can be easily extended to more general Lévy processes, as the ones treated in [7].

As discussed in Section 3, our results concerning the statistics of h-extrema of drifted Brownian motion correspond to the ones obtained in [8] via a non-rigorous Real Space Renormalization Group method applied to Sinai random walk with a vanishing bias. In addition to a rigorous derivation, here we are able to describe also the statistics of the h-slopes, lacking in [8].

In Section 5 (see Theorem 2), we analyze the behavior of the drifted Brownian motion around its h-extrema. While in the non-drifted case a generic h-slope not covering the origin behaves in proximity of its extremes as a 3D Bessel process, in the drifted case it behaves as a process with a cothangent drift, satisfying the SDE

$$\begin{cases} \mathrm{d}X_t = \mathrm{d}\beta_t \pm \mu \coth(\mu X_t)\,\mathrm{d}t, & t \geq 0, \\ X_0 = 0, \end{cases} \tag{1.1}$$

where β_t is a standard Brownian motion and the sign in the r.h.s. depends on the kind of h-slope (downward or upward) and on the kind of h-extrema (h-minimum or h-maximum). In addition, we show that the process (1.1) is simply the Brownian motion on $[0, \infty)$, starting at the origin, with drift $\pm\mu$, Doob-conditioned to hit $+\infty$ before 0.

The interest in the statistics of h-slopes and their behavior near the extremes comes also from the fact that, considering the diffusion in a drifted Brownian potential, the piecewise linear path obtained by connecting the h-extrema of the Brownian potential is the effective potential for the diffusion at large times [9].

2. Statistics of h-slopes of drifted Brownian motion

Given $\mu, x \in \mathbb{R}$ we denote by $\mathbf{P}_x^\mu$ the law on $C(\mathbb{R}, \mathbb{R})$ of the standard two-sided Brownian motion B with drift $-\mu$ having value x at time zero, i.e. $B_t = x + B_t^* - \mu t$ where $B^* : \mathbb{R} \to \mathbb{R}$

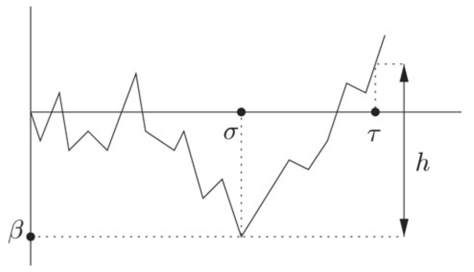

Fig. 1. The random variables β, σ, τ.

denotes the two-sided Brownian motion s.t. B_t has expectation zero and variance $|t|$. We denote the expectation w.r.t. $\mathbf{P}_x^\mu$ by $\mathbf{E}_x^\mu$. If $\mu = 0$ we simply write $\mathbf{P}_x, \mathbf{E}_x$.

Recall the definitions of h-maximum, h-minimum and h-extremum given in the Introduction. It is simple to verify that $\mathbf{P}_x^\mu$-a.s. the set of points of h-extrema is locally finite, unbounded from below and from above, and that points of h-minima alternate with points of h-maxima. The h-slope between two consecutive points of h-extrema α and β is defined as the truncated trajectory $\gamma := (B_t : t \in [\alpha, \beta])$. We call it an *upward* slope if α is a point of h-minimum (and consequently β is a point of h-maximum), otherwise we call it *downward* slope. The length $\ell(\gamma)$ and the height $h(\gamma)$ of the slope γ are defined as $\ell(\gamma) = \beta - \alpha$ and $h(\gamma) = |\gamma(\beta) - \gamma(\alpha)|$, respectively. Moreover, to the slope γ we associate the translated path $\theta(\gamma) := (B_{t+\alpha} - B_\alpha : t \in [0, \beta - \alpha])$. With some abuse of notation (as in Section 1) we call also $\theta(\gamma)$ the h-slope between the points of h-extrema α and β. When the context can cause some ambiguity, we will explicitly distinguish between the h-slope γ and the translated path $\theta(\gamma)$.

Finally, we introduce the following notation: given $\alpha \in \mathbb{R}$, the constant $\hat{\alpha}$ is defined as

$$\hat{\alpha} = \alpha + \mu^2/2. \tag{2.1}$$

2.1. The building blocks of the h-slopes

Given a two-sided Brownian B with law $\mathbf{P}_0^\mu$ we define the random variables b_t, τ, β, σ as follows (see Fig. 1):

$$\begin{cases} b_t = \min\{B_s : 0 \le s \le t\}, \\ \tau = \min\{t \ge 0 : B_t = b_t + h\}, \\ \beta = b_\tau = \min\{B_s : 0 \le s \le \tau\}, \\ \sigma = \max\{s : s \le \tau, \ B_s = \beta\}. \end{cases} \tag{2.2}$$

Note that $\mathbf{P}_0^\mu$-a.s. there exists a unique time $s \in [0, \tau]$ with $B_s = \beta$, which by definition coincides with σ.

Our analysis of the statistics of h-slopes for the drifted Brownian motion is based on the following lemma which extends to the drifted case the lemma in Section 1 of [1]:

Lemma 1. *Let $\mu \neq 0$, $\hat{\alpha} > 0$ and $x > 0$. Under $\mathbf{P}_0^\mu$, the two trajectories*

$$(B_t, 0 \le t \le \sigma), \qquad (B_{\sigma+t} - \beta, 0 \le t \le \tau - \sigma)$$

are independent; in particular (β, σ) and $\tau - \sigma$ are independent.

It holds

$$\mathbf{E}_0^\mu\left(\exp\left(-\alpha(\tau-\sigma)\right)\right)=\frac{\sqrt{2\hat{\alpha}}}{\mu}\frac{\sinh(\mu h)}{\sinh\left(\sqrt{2\hat{\alpha}}h\right)}. \tag{2.3}$$

Furthermore $-\beta$ *is exponentially distributed with mean*

$$\mathbf{E}_0^\mu(-\beta)=\frac{\sinh(\mu h)}{\mu e^{-\mu h}}, \tag{2.4}$$

and

$$\mathbf{E}_0^\mu\left[\exp(-\alpha\sigma)\mid\beta=-x\right]=\exp\left\{-x\left[\sqrt{2\hat{\alpha}}\coth\left(\sqrt{2\hat{\alpha}}h\right)-\mu\coth(\mu h)\right]\right\}. \tag{2.5}$$

In particular, $\mathbf{E}_0^\mu\left(\exp(-\alpha\sigma)\right)$ *is finite if and only if*

$$\sqrt{2\hat{\alpha}}\coth\left(\sqrt{2\hat{\alpha}}h\right)>\mu. \tag{2.6}$$

If (2.6) *is fulfilled, then*

$$\mathbf{E}_0^\mu\left(\exp(-\alpha\sigma)\right)=\frac{\mu e^{-\mu h}}{\sinh(\mu h)\left(\sqrt{2\hat{\alpha}}\coth\left(\sqrt{2\hat{\alpha}}h\right)-\mu\right)}. \tag{2.7}$$

The proof of the above lemma is based on excursion theory and identities concerning hitting times of the drifted Brownian motion. It will be given in Section 4.

2.2. *The probability measures* P_+^μ *and* P_-^μ *on h-slopes*

We define the path space $\mathcal{W}$ as the set

$$\mathcal{W}=\bigcup_{T\geq 0}C([0,T]).$$

Given $\gamma\in\mathcal{W}$, we define $\ell(\gamma)$ as the non-negative number such that $\gamma\in C[0,\ell(\gamma)]$ and we define the path $\gamma^*:[0,\infty)\to\mathbb{R}$ as

$$\gamma_t^*=\begin{cases}\gamma_t & \text{if } 0\leq t\leq\ell(\gamma),\\ \gamma_{\ell(\gamma)} & \text{if } t\geq\ell(\gamma).\end{cases}$$

Then the space $\mathcal{W}$ is a Polish space endowed with the metric $d_{\mathcal{W}}$ defined as

$$d_{\mathcal{W}}(\gamma_1,\gamma_2)=|\ell(\gamma_1)-\ell(\gamma_2)|+\|\gamma_1^*-\gamma_2^*\|_\infty.$$

On $\mathcal{W}$ we define the Borel probability measures P_+^μ, P_-^μ as follows. Let B, B' be independent Brownian motions with law $\mathbf{P}_0^\mu$. Recall definition (2.2) of τ,β,σ and define $b_t',\tau',\beta',\sigma'$ as (see Fig. 2):

$$\begin{cases}b_t'=\max\left\{B_s':0\leq s\leq t\right\},\\ \tau'=\min\left\{t\geq 0:B_t'=b_t'-h\right\},\\ \beta'=b_\tau'=\max\left\{B_s':0\leq s\leq\tau'\right\},\\ \sigma'=\max\left\{s:s\leq\tau',\ B_s'=\beta'\right\}.\end{cases} \tag{2.8}$$

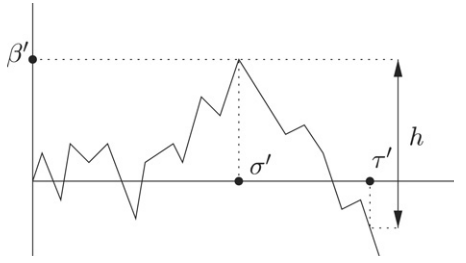

Fig. 2. The random variables β', σ', τ'.

Then P_+^μ is the law of the path γ, with $\ell(\gamma) = \tau - \sigma + \sigma'$, defined as

$$\gamma_t = \begin{cases} B_{\sigma+t} - \beta, & \text{if } t \in [0, \tau - \sigma], \\ B'_{t-(\tau-\sigma)} + h, & \text{if } t \in [\tau - \sigma, \tau - \sigma + \sigma'], \end{cases} \tag{2.9}$$

while P_-^μ is the law of the path γ, with $\ell(\gamma) = \tau' - \sigma' + \sigma$, defined as

$$\gamma_t = \begin{cases} B'_{\sigma'+t} - \beta', & \text{if } t \in [0, \tau' - \sigma'], \\ B_{t-(\tau'-\sigma')} - h, & \text{if } t \in [\tau' - \sigma', \tau' - \sigma' + \sigma]. \end{cases} \tag{2.10}$$

Note that P_-^μ equals the law of the path $-\gamma$ if γ is chosen with law $P_+^{-\mu}$.

We introduce two disjoint subsets $\mathcal{W}_+$ and $\mathcal{W}_-$ of $\mathcal{W}$:

$$\begin{aligned} \mathcal{W}_+ &= \{\gamma \in \mathcal{W} : \gamma_0 = \min\{\gamma_t : t \in [0, \ell(\gamma)]\} = 0, \\ &\qquad \gamma_{\ell(\gamma)} = \max\{\gamma_t : t \in [0, \ell(\gamma)]\} \geq h\Big\}, \\ \mathcal{W}_- &= \{\gamma \in \mathcal{W} : \gamma_0 = \max\{\gamma_t : t \in [0, \ell(\gamma)]\} = 0, \\ &\qquad \gamma_{\ell(\gamma)} = \min\{\gamma_t : t \in [0, \ell(\gamma)]\} \leq -h\Big\}. \end{aligned}$$

Then the probability measure $P_\pm^\mu$ is concentrated on $\mathcal{W}_\pm$. Below we will prove that, given the two-sided BM with law $\mathbf{P}_0^\mu$, P_+^μ is the law of the generic upward h-slope not covering the origin, while P_-^μ is the law of the generic downward h-slope not covering the origin.

We collect in what follows some results derived from Lemma 1 and straightforward computations, which will be useful in what follows:

Proposition 1. *Fix $\mu \neq 0$. Let (ℓ_+, ζ_+) be the random vector distributed as $\left(\ell(\gamma), \gamma_{\ell(\gamma)} - h\right)$ where γ is chosen with law P_+^μ and let (ℓ_-, ζ_-) be the random vector distributed as $\left(\ell(\gamma), -(\gamma_{\ell(\gamma)} + h)\right)$ where γ is chosen with law P_-^μ.*

Then ζ_+, ζ_- are exponential variables of mean

$$\mathbf{E}(\zeta_+) = \frac{\sinh(\mu h)}{\mu \mathrm{e}^{\mu h}}, \qquad \mathbf{E}(\zeta_-) = \frac{\sinh(\mu h)}{\mu \mathrm{e}^{-\mu h}}. \tag{2.11}$$

Fix α such that $\hat{\alpha} := \alpha + \mu^2/2 > 0$. Then for all $x > 0$

$$\begin{aligned} \mathbf{E}\left(\mathrm{e}^{-\alpha \ell_\pm} \mid \zeta_\pm = x\right) = \frac{\sqrt{2\hat{\alpha}}}{\mu} \frac{\sinh(\mu h)}{\sinh\left(\sqrt{2\hat{\alpha}} h\right)} \\ \times \exp\left\{-x\left(\sqrt{2\hat{\alpha}} \coth\left(\sqrt{2\hat{\alpha}} h\right) - \mu \coth(\mu h)\right)\right\}. \end{aligned} \tag{2.12}$$

In particular, the expectation $\mathbf{E}\left(e^{-\alpha\ell_\pm-\lambda\zeta_\pm}\right)$ *is finite if and only if*

$$\sqrt{2\hat{\alpha}}\coth\left(\sqrt{2\hat{\alpha}}h\right)+(\lambda\pm\mu)>0. \tag{2.13}$$

If (2.13) *is fulfilled, then*

$$\mathbf{E}\left(e^{-\alpha\ell_\pm-\lambda\zeta_\pm}\right)=\frac{\sqrt{2\hat{\alpha}}e^{\pm\mu h}}{\sqrt{2\hat{\alpha}}\cosh\left(\sqrt{2\hat{\alpha}}h\right)+(\lambda\pm\mu)\sinh\left(\sqrt{2\hat{\alpha}}h\right)}. \tag{2.14}$$

Hence

$$\mathbf{E}(\ell_+)=\mu^{-2}\left(\mu h-\sinh(\mu h)e^{-\mu h}\right), \tag{2.15}$$

$$\mathbf{E}(\ell_-)=\mu^{-2}\left(e^{\mu h}\sinh(\mu h)-\mu h\right). \tag{2.16}$$

Consider $\ell:=\ell_-+\ell_+$, *where* ℓ_- *and* ℓ_+ *are chosen independently. Then,*

$$\mathbf{E}(\ell)=\frac{2}{\mu^2}\sinh^2(\mu h). \tag{2.17}$$

Given $\alpha\in\mathbb{R}$, $\mathbf{E}\left(e^{-\alpha\ell}\right)$ *is finite if and only if*

$$\begin{cases}\hat{\alpha}:=\alpha+\mu^2/2>0,\\ 2\alpha\cosh^2(\sqrt{2\hat{\alpha}}h)+\mu^2>0.\end{cases} \tag{2.18}$$

If (2.18) *is fulfilled, then*

$$\mathbf{E}\left(e^{-\alpha\ell}\right)=\frac{2\hat{\alpha}}{2\alpha\cosh^2(\sqrt{2\hat{\alpha}}h)+\mu^2}. \tag{2.19}$$

Remark 1. Due to the identity

$$2\alpha\cosh^2(\sqrt{2\hat{\alpha}}h)+\mu^2=\frac{\cosh^2(\sqrt{2\hat{\alpha}}h)}{h^2}\left(2\hat{\alpha}h^2-\mu^2h^2\tanh^2(\sqrt{2\hat{\alpha}}h)\right),$$

by straightforward computations one can check that for $\mu h>1$ condition (2.18) is fulfilled if and only if $\alpha>-\mu^2/2+y_*^2/(2h^2)$, where y_* is the only positive solution of the equation $y=\mu h\tanh(y)$. If $\mu h\le 1$ then condition (2.18) is fulfilled if and only if $\alpha>-\mu^2/2$.

2.3. The stationary alternating marked simple point process $\mathcal{P}^\mu$

We denote by $\mathcal{N}$ the space of sequences $\xi=\{(x_i,\gamma_i):i\in\mathbb{Z}\}$ such that (1) $(x_i,\gamma_i)\in\mathbb{R}\times\mathcal{W}$, (2) $x_i<x_{i+1}$, (3) $x_0\le 0<x_1$ and (4) $\lim_{i\to\pm\infty}x_i=\pm\infty$. In what follows, ξ will be often identified with the counting measure $\sum_{i\in\mathbb{Z}}\delta_{(x_i,\gamma_i)}$ on $\mathbb{R}\times\mathcal{W}$.

$\mathcal{N}$ is a measurable space with σ-algebra of measurable sets generated by

$$\{\xi\in\mathcal{N}:\xi(A\times B)=j\},\quad A\subset\mathbb{R}\text{ Borel}, B\subset\mathcal{W}\text{ Borel}, j\in\mathbb{N}.$$

One can characterize the above σ-algebra as follows. Consider the space $\mathcal{S}:=(0,\infty)\times(0,\infty)^{\mathbb{Z}}\times\mathcal{W}^{\mathbb{Z}}$ as a measurable space with σ-algebra of measurable sets given by the Borel subsets associated to the product topology. Call $\mathcal{S}'$ the subset of $\mathcal{S}$ given by the elements where the first entry is not larger than the entry with index 0 of the factor space $(0,\infty)^{\mathbb{Z}}$. Then by the same arguments leading to [2] [Proposition 7.1.X] one can prove that the map

$$\mathcal{N} \ni \{(x_i,\gamma_i)\}_{i\in\mathbb{Z}} \to \{x_1\}\times\{\tau_i\}_{i\in\mathbb{Z}}\times\{\gamma_i\}_{i\in\mathbb{Z}} \in \mathcal{S}', \quad \tau_i := x_{i+1}-x_i, \tag{2.20}$$

is bijective and both ways measurable. We note that the introduction of $\mathcal{S}'$ is due to the constraint $x_1 \le \tau_0$.

Let us define $\mathcal{P}^{\mu}_{0,\pm}$ as the law of the sequences $\{(x_i,\gamma_i)\}_{i\in\mathbb{Z}} \in \mathcal{N}$ such that $\{\gamma_i\}_{i\in\mathbb{Z}}$ are independent random paths, $\{\gamma_{2i}\}_{i\in\mathbb{Z}}$ are i.i.d. random paths with law $P^{\mu}_{\pm}$, $\{\gamma_{2i+1}\}_{i\in\mathbb{Z}}$ are i.i.d. random paths with law $P^{\mu}_{\mp}$, $x_0=0$ and $x_{i+1}-x_i=\ell(\gamma_i)$. Note that $\mathcal{P}^{\mu}_{0,\pm}$ is concentrated on the measurable subset $\mathcal{N}_0$ defined as

$$\mathcal{N}_0 = \{\{(x_i,\gamma_i)\}_{i\in\mathbb{Z}} : x_0 = 0\}.$$

Finally we consider the convex combination

$$\mathcal{P}^{\mu}_0 = \frac{1}{2}\mathcal{P}^{\mu}_{0,+} + \frac{1}{2}\mathcal{P}^{\mu}_{0,-}.$$

Let $\theta : \mathcal{N}\to\mathcal{N}$ and, for all $t\in\mathbb{R}$, let $T_t : \mathcal{N}\to\mathcal{N}$ be the maps defined as

$$\theta\xi = \sum_{i\in\mathbb{Z}} \delta_{(x_i-x_1,\gamma_i)}, \qquad T_t\xi = \sum_{i\in\mathbb{Z}} \delta_{(x_i+t,\gamma_i)} \quad \text{if } \xi = \sum_{i\in\mathbb{Z}} \delta_{(x_i,\gamma_i)}.$$

We stress that the above translation map T_t coincides with the map T_{-t} of [3] and with the map S_{-t} of [2]. A probability measure $\mathcal{Q}$ on $\mathcal{N}$ is called stationary if $T_t\mathcal{Q}(A) := \mathcal{Q}(T_tA) = \mathcal{Q}(A)$ for all $t\in\mathbb{R}$ and all $A\subset\mathcal{N}$ measurable, while it is called θ-invariant if $\theta\mathcal{Q}(A) := \mathcal{Q}(\theta A) = \mathcal{Q}(A)$ for all $A\subset\mathcal{N}$ measurable.

Note that $\mathcal{P}^{\mu}_0$ is θ-invariant. Moreover, due to (2.17) in Lemma 1,

$$\mathbf{E}_{\mathcal{P}^{\mu}_0}(x_1) = \mathbf{E}_{\mathcal{P}^{\mu}_0}(\ell(\gamma_0)) = \mathbf{E}(\ell)/2 = \sinh^2(\mu h)/\mu^2. \tag{2.21}$$

Hence, due to the Palm–Khinchin theory (see Theorem 1.3.1 and formula (1.2.15) in [3], and Theorem 12.3.II in [2]) there exists a unique stationary measure $\mathcal{P}^{\mu}$ on $\mathcal{N}$ such that

$$\begin{aligned}\mathcal{P}^{\mu}(A) &= \frac{2}{\mathbf{E}(\ell)}\mathbf{E}_{\mathcal{P}^{\mu}_0}\left[\int_0^{x_1} \chi\left(T_{-t}\left(\{x_i,\gamma_i\}_{i\in\mathbb{Z}}\right)\in A\right)\mathrm{d}t\right],\\ &= \frac{\mu^2}{\sinh^2(\mu h)}\mathbf{E}_{\mathcal{P}^{\mu}_0}\left[\int_0^{x_1} \chi\left(T_{-t}\left(\{x_i,\gamma_i\}_{i\in\mathbb{Z}}\right)\in A\right)\mathrm{d}t\right],\end{aligned} \tag{2.22}$$

where $\chi(\cdot)$ denotes the characteristic function. We simply say that $\mathcal{P}^{\mu}$ *is the law of the stationary alternating marked simple point process on* $\mathbb{R}$ *with alternating mark laws given by* P^{μ}_{+}, P^{μ}_{-}. The probability measure $\mathcal{P}^{\mu}_0$ is the so-called *Palm distribution* associated to $\mathcal{P}^{\mu}$.

One can write

$$\mathcal{P}^{\mu}(\cdot) = \mathcal{P}^{\mu}(\cdot \mid \gamma_0\in\mathcal{W}_+)\mathcal{P}^{\mu}(\gamma_0\in\mathcal{W}_+) + \mathcal{P}^{\mu}(\cdot \mid \gamma_0\in\mathcal{W}_-)\mathcal{P}^{\mu}(\gamma_0\in\mathcal{W}_-). \tag{2.23}$$

From (2.22) we obtain that

$$\mathcal{P}^{\mu}(\gamma_0\in\mathcal{W}_\pm) = \frac{2\mathbf{E}_{\mathcal{P}^{\mu}_0}\left[x_1\chi(\gamma_0\in\mathcal{W}_\pm)\right]}{\mathbf{E}(\ell)} = \frac{\mathbf{E}_{P^{\mu}_{\pm}}(\ell(\gamma))}{\mathbf{E}(\ell)} = \frac{\mathbf{E}(\ell_\pm)}{\mathbf{E}(\ell)}. \tag{2.24}$$

Hence, by (2.15) and (2.16) of Lemma 1, we can conclude that

$$\mathcal{P}^{\mu}(\gamma_0 \in \mathcal{W}_{\pm}) = \frac{\pm\mu h \mp \sinh(\mu h)\mathrm{e}^{\mp\mu h}}{2\sinh^2(\mu h)}. \tag{2.25}$$

In order to describe the conditional probability measure $\mathcal{P}^{\mu}(\cdot \mid \gamma_0 \in \mathcal{W}_{\pm})$ we first observe that x_1 and $\{\gamma_i\}_{i\in\mathbb{Z}}$ univocally determine the set $\{(x_i, \gamma_i)\}_{i\in\mathbb{Z}}$. Moreover from (2.22) we derive that, given Borel subsets $A \subset \mathbb{R}$, $B_j \subset \mathcal{W}$ for $-m \le j \le n$, it holds

$$\begin{aligned}
&\mathcal{P}^{\mu}(x_1 \in A, \gamma_j \in B_j\ \forall j : -m \le j \le n \mid \gamma_0 \in \mathcal{W}_{\pm})\\
&\quad = \mathbf{E}_{\mathcal{P}^{\mu}_{0,\pm}}\left(\int_0^{x_1} \chi(x_1 - t \in A)\mathrm{d}t, \gamma_j \in B_j\ \forall j : -m \le j \le n\right)/\mathbf{E}(\ell_{\pm})\\
&\quad = \mathbf{E}_{P^{\mu}_{\pm}}\left(\int_0^{\ell(\gamma)} \chi(t \in A)\mathrm{d}t, \gamma \in B_0\right)\\
&\qquad \times \left[\prod_{\substack{-m\le j\le n\\ j\neq 0,\ \text{odd}}} P^{\mu}_{\mp}(B_j)\right]\left[\prod_{\substack{-m\le j\le n\\ j\neq 0,\ \text{even}}} P^{\mu}_{\pm}(B_j)\right]/\mathbf{E}(\ell_{\pm}).
\end{aligned}$$

This identity implies that under $\mathcal{P}^{\mu}(\cdot \mid \gamma_0 \in \mathcal{W}_{\pm})$ the random paths γ_i, $i \in \mathbb{Z}$, are independent, the paths $\{\gamma_{2i}\}_{i\in\mathbb{Z}\setminus\{0\}}$ have common law $P^{\mu}_{\pm}$ while the paths $\{\gamma_{2i+1}\}_{i\in\mathbb{Z}}$ have common law $P^{\mu}_{\mp}$ and that the path γ_0 has law

$$\mathcal{P}^{\mu}(\gamma_0 \in A \mid \gamma_0 \in \mathcal{W}_{\pm}) = \frac{\mathbf{E}_{P^{\mu}_{\pm}}(\ell(\gamma)\chi(\gamma \in A))}{\mathbf{E}_{P^{\mu}_{\pm}}(\ell(\gamma))} = \frac{\mathbf{E}_{P^{\mu}_{\pm}}(\ell(\gamma)\chi(\gamma \in A))}{\mathbf{E}(\ell_{\pm})}. \tag{2.26}$$

Finally, we claim that under $\mathcal{P}^{\mu}(\cdot \mid \gamma_0 \in \mathcal{W}_{\pm})$, x_1 has probability density function on $[0, \infty)$ given by $(1 - F_{\pm}(x))/\mathbf{E}(\ell_{\pm})$, where $F_{\pm}(x) = \mathbf{P}(\ell_{\pm} \le x)$ for $x \ge 0$. Indeed, given $x \ge 0$, due to (2.22) and (2.24) we can write

$$\begin{aligned}
&\mathcal{P}^{\mu}(x_1 \ge x \mid \gamma_0 \in \mathcal{W}_{\pm}) = \mathcal{P}^{\mu}(x_1 \ge x, \gamma_0 \in \mathcal{W}_{\pm})/\mathcal{P}^{\mu}(\gamma_0 \in \mathcal{W}_{\pm})\\
&\quad = \mathbf{E}_{\mathcal{P}^{\mu}_{0,\pm}}\left(\int_0^{x_1} \chi(x_1 - t \ge x)\mathrm{d}t\right)/\mathbf{E}(\ell_{\pm}) = \mathbf{E}\left((\ell_{\pm} - x)\chi(\ell_{\pm} \ge x)\right)/\mathbf{E}(\ell_{\pm})\\
&\quad = \mathbf{E}\left(\int_x^{\infty} \chi(\ell_{\pm} \ge z)\mathrm{d}z\right)/\mathbf{E}(\ell_{\pm}) = \left(\int_x^{\infty} \mathbf{P}(\ell_{\pm} \ge z)\mathrm{d}z\right)/\mathbf{E}(\ell_{\pm}).
\end{aligned}$$

In particular, we point out that due to (2.23) and (2.24) under $\mathcal{P}^{\mu}$ the random variable x_1 has probability density on $[0, \infty)$ given by $(2 - F_{+}(x) - F_{-}(x))/\mathbf{E}(\ell)$.

2.4. *The statistics of h-slopes*

We can finally prove the main theorem of this section:

Theorem 1. *Let B be the Brownian motion with law $\mathbf{P}^{\mu}_0$ and let $(\mathfrak{m}_i)_{i\in\mathbb{Z}}$ be the sequence of points of h-extrema of B, increasingly ordered, with $\mathfrak{m}_0 \le 0 < \mathfrak{m}_1$. For each $i \in \mathbb{Z}$ define the h-slope γ_i as*

$$\gamma_i = \left(B_t - B_{\mathfrak{m}_i} : 0 \le t \le \mathfrak{m}_{i+1} - \mathfrak{m}_i\right).$$

Then the marked point process $\{(\mathfrak{m}_i, \gamma_i) : i \in \mathbb{Z}\}$ has law $\mathcal{P}^{\mu}$.

In order to prove the above result we need to further elucidate the relation between $\mathcal{P}^\mu$ and its Palm distribution $\mathcal{P}_0^\mu$, benefiting from the Palm–Khinchin theory. To this end, we need the ergodicity of $\mathcal{P}_0^\mu$:

Lemma 2. *The probability measure $\mathcal{P}_0^\mu$ is θ-ergodic, i.e. if $A \subset \mathcal{N}$ is a measurable set such that $\mathcal{P}_0^\mu(A \Delta \theta A) = 0$ then $\mathcal{P}_0^\mu(A) \in \{0, 1\}$.*

Proof. Let us suppose that $A \subset \mathcal{N}$ is a Borel set with $\mathcal{P}_0^\mu(A \Delta \theta A) = 0$. Due to the characterization of the σ-algebra of measurable sets in $\mathcal{N}$ given by the bijective and both ways measurable map (2.20) and since $\mathcal{P}_0^\mu$ is concentrated on $\mathcal{N}_0$, for each $\varepsilon > 0$ we can find a measurable set $A_\varepsilon \subset \mathcal{N}_0$ and an integer $k = k(\varepsilon)$ such that A_ε depends only on the random variables γ_i with $-k \le i \le k$ and $\mathcal{P}_0^\mu(A \Delta A_\varepsilon) \le \varepsilon$. This implies that $\mathcal{P}_{0,+}^\mu(A \Delta A_\varepsilon) \le 2\varepsilon$ and $\mathcal{P}_{0,-}^\mu(A \Delta A_\varepsilon) \le 2\varepsilon$. Since $\mathcal{P}_0^\mu$ is θ-invariant and $\mathcal{P}_0^\mu(A \Delta \theta A) = 0$, we get for each positive integer n that $\mathcal{P}_0^\mu(A \Delta \theta^n A) = 0$ and therefore

$$\mathcal{P}_0^\mu(A \Delta \theta^n A_\varepsilon) = \mathcal{P}_0^\mu(\theta^n A \Delta \theta^n A_\varepsilon) = \mathcal{P}_0^\mu\left(\theta^n (A \Delta A_\varepsilon)\right) = \mathcal{P}_0^\mu(A \Delta A_\varepsilon) \le \varepsilon.$$

This implies that

$$\mathcal{P}_0^\mu(A_\varepsilon \Delta \theta^n A_\varepsilon) \le \mathcal{P}_0^\mu(A_\varepsilon \Delta A) + \mathcal{P}_0^\mu(A \Delta \theta^n A_\varepsilon) \le 2\varepsilon. \tag{2.27}$$

Let us now write $o(1)$ for any quantity which goes to 0 as $\varepsilon \downarrow 0$. We note that for n large enough and even it holds

$$\mathcal{P}_{0,+}^\mu(A_\varepsilon \cap \theta^n A_\varepsilon) = \mathcal{P}_{0,+}^\mu(A_\varepsilon)^2 = \mathcal{P}_{0,+}^\mu(A)^2 + o(1), \tag{2.28}$$

$$\mathcal{P}_{0,-}^\mu(A_\varepsilon \cap \theta^n A_\varepsilon) = \mathcal{P}_{0,-}^\mu(A_\varepsilon)^2 = \mathcal{P}_{0,-}^\mu(A)^2 + o(1); \tag{2.29}$$

while for n large enough and odd it holds

$$\begin{aligned} \mathcal{P}_{0,+}^\mu(A_\varepsilon \cap \theta^n A_\varepsilon) = \mathcal{P}_{0,-}^\mu(A_\varepsilon \cap \theta^n A_\varepsilon) &= \mathcal{P}_{0,+}^\mu(A_\varepsilon)\mathcal{P}_{0,-}^\mu(A_\varepsilon) \\ &= \mathcal{P}_{0,+}^\mu(A)\mathcal{P}_{0,-}^\mu(A) + o(1). \end{aligned} \tag{2.30}$$

Let $a := \mathcal{P}_{0,+}^\mu(A)$ and $b := \mathcal{P}_{0,-}^\mu(A)$. Due to our previous observations, we conclude that

$$\begin{aligned} o(1) &= \mathcal{P}_0^\mu(A_\varepsilon \Delta \theta^n A_\varepsilon) \\ &= \mathcal{P}_0^\mu(A_\varepsilon) + \mathcal{P}_0^\mu(\theta^n A_\varepsilon) - 2\mathcal{P}_0^\mu(A_\varepsilon \cap \theta^n A_\varepsilon) \\ &= 2\mathcal{P}_0^\mu(A_\varepsilon) - 2\mathcal{P}_0^\mu(A_\varepsilon \cap \theta^n A_\varepsilon) \\ &= \mathcal{P}_{0,+}^\mu(A_\varepsilon) + \mathcal{P}_{0,-}^\mu(A_\varepsilon) - \left[\mathcal{P}_{0,+}^\mu(A_\varepsilon \cap \theta^n A_\varepsilon) + \mathcal{P}_{0,-}^\mu(A_\varepsilon \cap \theta^n A_\varepsilon)\right] \\ &= \begin{cases} a + b - (a^2 + b^2) + o(1) & \text{if } n \text{ is even and large,} \\ a + b - 2ab + o(1) & \text{if } n \text{ is odd and large.} \end{cases} \end{aligned} \tag{2.31}$$

It is simple to check that there are only two possible cases: (i) $a = o(1)$ and $b = o(1)$, (ii) $a = 1 + o(1)$ and $b = 1 + o(1)$, implying respectively that $\mathcal{P}_0^\mu(A) = o(1)$ and $\mathcal{P}_0^\mu(A) = 1 + o(1)$. Due to the arbitrariness of ε we conclude that $\mathcal{P}_0^\mu(A) \in \{0, 1\}$. □

Let us now introduce the space $\mathcal{N}_*$ given by the counting measures $\xi = \sum_{j \in J} n_j \delta_{(x^{(j)}, \gamma^{(j)})}$ on $\mathbb{R} \times \mathcal{W}$, where $n_j \in \mathbb{N}$ and the set $\{\left(x^{(j)}, \gamma^{(j)}\right)_{j \in J}\}$ has finite intersection with sets of the form $[a, b] \times \mathcal{W}$. Note that if $n_j = 1$ for all $j \in J$ we can identity ξ with its support. Hence we can think of $\mathcal{N}$ as a subset of $\mathcal{N}_*$.

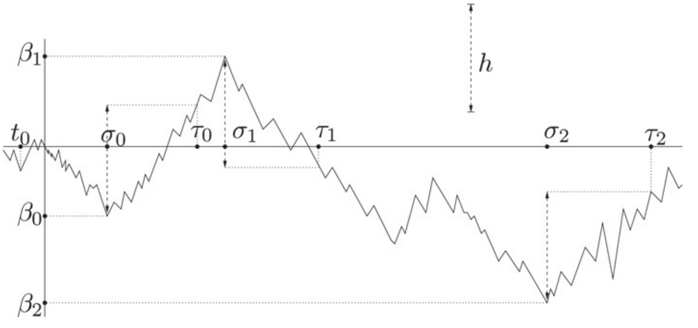

Fig. 3. The sequence σ_n, τ_n.

As discussed in [3] [Section 1.1.5] one defines on $\mathcal{N}_*$ a suitable metric $d_{\mathcal{N}_*}$ such that (i) $\mathcal{N}_*$ is a Polish space, (ii) $\mathcal{N}$ is a Borel subset of $\mathcal{N}_*$ and (iii) the σ-algebra of Borel subsets of $\mathcal{N}_*$ is generated by the sets $\{\xi \in \mathcal{N}_* : \xi(A \times B) = j\}$, where $j \in \mathbb{N}$, $A \subset \mathbb{R}$ and $B \subset \mathcal{W}$ are Borel subsets. In particular, the σ-algebra of measurable subsets of $\mathcal{N}$ introduced above coincides with the σ-algebra of Borel subsets of $\mathcal{N}$ and we can think of $\mathcal{P}^\mu, \mathcal{P}_0^\mu, \mathcal{P}_{0,\pm}^\mu$ as Borel probability measures on $\mathcal{N}_*$ concentrated on $\mathcal{N}$.

Due to Lemma 2 and [3] [Theorem 1.3.13] we get

Corollary 1. *Given $\mu \neq 0$, the probability measure $T_{t_0}\mathcal{P}_0^\mu$ weakly converges to $\mathcal{P}^\mu$ as $t_0 \downarrow -\infty$, i.e. for any continuous bounded function $f : \mathcal{N}_* \to \mathbb{R}$ it holds*

$$\lim_{t_0 \downarrow -\infty} \mathbf{E}_{T_{t_0}\mathcal{P}_0^\mu}(f) = \mathbf{E}_{\mathcal{P}^\mu}(f). \tag{2.32}$$

Let ν be the intensity measure associated to $\mathcal{P}^\mu$, i.e. ν is the probability measure on $\mathbb{R} \times \mathcal{W}$ such that

$$\nu(A \times B) = \mathbf{E}_{\mathcal{P}^\mu}\left(\xi(A \times B)\right), \quad A \subset \mathbb{R} \text{ Borel}, B \subset \mathcal{W} \text{ Borel}.$$

Then due to [3] [Theorem 1.1.16], the weak convergence of $T_{t_0}\mathcal{P}_0^\mu$ to $\mathcal{P}^\mu$ stated in Corollary 1 is equivalent to the following fact: given a finite family $X_1, X_2, \ldots, X_k$ of disjoint sets $X_j = [a_j, b_j) \times L_j$, with $a_j, b_j \in \mathbb{R}$ and $L_j \subset \mathcal{W}$ Borel, satisfying $\nu(\partial X_j) = 0$ for $j = 1, \ldots, k$, it holds

$$\lim_{t_0 \downarrow -\infty} T_{t_0}\mathcal{P}_0^\mu\left(\xi(X_s) = j_s \ \forall s : 1 \le s \le k\right) = \mathcal{P}^\mu\left(\xi(X_s) = j_s \ \forall s : 1 \le s \le k\right),$$

for all $j_1, j_2, \ldots, j_k \in \mathbb{N}$.

We have now the main tools in order to prove Theorem 1.

Proof of Theorem 1. Let B be a two-sided Brownian motion with law $\mathbf{P}_0^\mu$. Set $\tau_{-1} = t_0$ and define the random variables $\tau_n, \beta_n, \sigma_n$ inductively on $n \in \mathbb{N}$ as follows (see Fig. 3):

For n even set

$$\tau_n = \min\left\{t \ge \tau_{n-1} : B_t = \min_{\tau_{n-1} \le s \le t}(B_s) + h\right\}, \tag{2.33}$$

$$\beta_n = \min\left\{B_s : \tau_{n-1} \le s \le \tau_n\right\}, \tag{2.34}$$

$$\sigma_n = \max\left\{s : \tau_{n-1} \le s \le \tau_n, B_s = \beta_n\right\}; \tag{2.35}$$

for n odd set

$$\tau_n = \min\left\{ t \geq \tau_{n-1} : B_t = \max_{\tau_{n-1} \leq s \leq t}(B_s) - h \right\}, \tag{2.36}$$

$$\beta_n = \max\{B_s : \tau_{n-1} \leq s \leq \tau_n\}, \tag{2.37}$$

$$\sigma_n = \max\{s : \tau_{n-1} \leq s \leq \tau_n, B_s = \beta_n\}. \tag{2.38}$$

Note that by construction σ_n is a point of h-maximum for n odd, while σ_n is a point of h-minimum for $n \neq 0$ even. Moreover, due to Lemma 1 and the strong Markov property of Brownian motion at the Markov times τ_n, the slopes

$$\left(B_{\sigma_n+s} - B_{\sigma_n} : 0 \leq s \leq \sigma_{n+1} - \sigma_n\right) \quad n \geq 1$$

are independent, having law P_-^μ if n is odd and law P_+^μ if n is even.

In what follows we will use the independent random variables X, V with the following laws: X is distributed as $\sigma_1 - t_0$, i.e. X is distributed as $\bar\tau + \bar\sigma'$ where $\bar\tau, \bar\sigma'$ are independent copies of τ, σ' defined in (2.2) and (2.8) respectively. V is distributed as ℓ_+, i.e. as the length $\ell(\gamma)$ of the random path γ chosen with law P_+^μ.

Given a realization of the two-sided Brownian motion B with law $\mathbf{P}_0^\mu$, let $\xi(B)$ be the associated marked simple point process defined at the end of Theorem 1, while let ξ denote a generic element of $\mathcal{N}$. Fix a finite family $X_1, X_2, \ldots, X_k$ of disjoint sets $X_j = [a_j, b_j) \times L_j$, with $a_i, b_j \in \mathbb{R}$ and $L_j \subset \mathcal{W}$ Borel, and consider the event $\mathcal{A} := \{\xi : \xi(X_m) = j_m,\ 1 \leq m \leq k\}$ for given $j_1, j_2, \ldots, j_k \in \mathbb{N}$. Finally, set $a := \min\{a_1, a_2, \ldots, a_k\}$. Due to the discussion after Corollary 1, we only need to show that $\mathbf{P}_0^\mu(\xi(B) \in \mathcal{A}) = \mathcal{P}^\mu(\mathcal{A})$. To this end, we set $g(u) := T_u \mathcal{P}_{0,-}^\mu(\mathcal{A})$ and restrict in what follows to the case $t_0 < a$. Then our initial considerations imply that

$$\mathbf{P}_0^\mu(\xi(B) \in \mathcal{A}, \sigma_1 < a) = \mathbf{E}(g(t_0 + X), t_0 + X < a) \tag{2.39}$$

and therefore

$$\begin{aligned} &\left|\mathbf{P}_0^\mu(\xi(B) \in \mathcal{A}) - \mathbf{E}g(t_0 + X)\right| \\ &\quad = \left|\mathbf{P}_0^\mu(\xi(B) \in \mathcal{A},\ \sigma_1 \geq a) - \mathbf{E}(g(t_0 + X),\ t_0 + X \geq a)\right| \\ &\quad \leq 2P(t_0 + X \geq a). \end{aligned} \tag{2.40}$$

In what follows, we will frequently apply the above argument in order to get estimates from above without explicit mention.

Let us consider the probability measure $T_{t_0}\mathcal{P}_0^\mu$. By definition

$$T_{t_0}\mathcal{P}_0^\mu(\mathcal{A}) = \frac{1}{2} T_{t_0}\mathcal{P}_{0,+}^\mu(\mathcal{A}) + \frac{1}{2} T_{t_0}\mathcal{P}_{0,-}^\mu(\mathcal{A}),$$

while

$$\left|T_{t_0}\mathcal{P}_{0,+}^\mu(\mathcal{A}) - \mathbf{E}g(t_0 + V)\right| \leq 2P(t_0 + V \geq a). \tag{2.41}$$

Hence we can estimate

$$\left|T_{t_0}\mathcal{P}_0^\mu(\mathcal{A}) - \mathbf{E}g(t_0 + V)/2 - g(t_0)/2\right| \leq 2P(t_0 + V \geq a). \tag{2.42}$$

Due to (2.40) and (2.42) and Corollary 1, in order to prove the theorem it is enough to show that

$$\lim_{t_0 \downarrow -\infty} |\mathbf{E}g(t_0 + X) - \mathbf{E}g(t_0 + V)/2 - g(t_0)/2| = 0. \tag{2.43}$$

We claim that, given a generic positive random variable W having a (bounded) probability density and bounded third moment, $|\mathbf{E}g(t_0 + W) - g(t_0)|$ converges to 0 as $t_0 \downarrow -\infty$. Due to Lemma 3, this result allows deriving (2.43). In order to prove our claim, define S_n as the sum of n independent copies of the random variable ℓ introduced in Proposition 1. Moreover, let S_n and W be independent. Then

$$\begin{aligned}
&|\mathbf{E}g(t_0 + W + S_n) - \mathbf{E}g(t_0 + W)| \le 2P(t_0 + W + S_n \ge a),\\
&|\mathbf{E}g(t_0 + S_n) - \mathbf{E}g(t_0)| \le 2P(t_0 + S_n \ge a).
\end{aligned}$$

Hence in order to prove our claim, it is enough to prove that

$$\lim_{n\uparrow\infty} \limsup_{t_0\downarrow-\infty} |\mathbf{E}g(t_0 + W + S_n) - \mathbf{E}g(t_0 + S_n)| = 0.$$

In general, given a r.v. Zwe write p_Z for its probability density (if it exists). Moreover, we denote by $\|\cdot\|_1$ the norm in $L^1(\mathbb{R}, \mathrm{d}u)$. Setting $\bar{S}_n = S_n - n\mathbf{E}(\ell)$, we can bound

$$\begin{aligned}
&|\mathbf{E}g(t_0 + W + S_n) - \mathbf{E}g(t_0 + S_n)|\\
&\quad \le \int_{\mathbb{R}} g(t_0 + u) \left| p_{W+S_n}(u) - p_{S_n}(u) \right| \mathrm{d}u \le \| p_{W+\bar{S}_n} - p_{\bar{S}_n} \|_1.
\end{aligned} \tag{2.44}$$

Since

$$\begin{aligned}
&p_{W+\bar{S}_n}(u)\mathrm{d}u = p_{(W+\bar{S}_n)/\sqrt{n}}(u/\sqrt{n})\mathrm{d}(u/\sqrt{n}),\\
&p_{\bar{S}_n}(u)\mathrm{d}u = p_{\bar{S}_n/\sqrt{n}}(u/\sqrt{n})\mathrm{d}(u/\sqrt{n}),
\end{aligned}$$

by a change of variables we can conclude that

$$|\mathbf{E}g(t_0 + W + S_n) - \mathbf{E}g(t_0 + S_n)| \le \| p_{(W+\bar{S}_n)/\sqrt{n}} - p_{\bar{S}_n}/\sqrt{n} \|_1. \tag{2.45}$$

Let $\mathcal{N}(u) = \exp\left(-u^2/(2\lambda)\right)/\sqrt{2\pi\lambda}$ be the Gaussian distribution with variance $\lambda = \mathrm{Var}(\ell)$. Due to (2.44) and (2.45) in order to conclude we only need to show that

$$\lim_{n\uparrow\infty} \| p_{(W+\bar{S}_n)/\sqrt{n}} - p_{W/\sqrt{n}} * \mathcal{N} \|_1 = 0, \tag{2.46}$$

$$\lim_{n\uparrow\infty} \| p_{W/\sqrt{n}} * \mathcal{N} - \mathcal{N} \|_1 = 0, \tag{2.47}$$

$$\lim_{n\uparrow\infty} \| \mathcal{N} - p_{\bar{S}_n/\sqrt{n}} \|_1 = 0, \tag{2.48}$$

where $f * g$ denotes the convolution of f and g.

We note that (2.46) follows from (2.48) since $p_{(W+\bar{S}_n)/\sqrt{n}} = p_{W/\sqrt{n}} * p_{\bar{S}_n/\sqrt{n}}$ and for generic functions h, h', w in $L^1(\mathbb{R}, \mathrm{d}u)$ it holds that $\|h * w - h' * w\|_1 \le \|h - h'\|_1 \|w\|_1$. Limit (2.47) follows from straightforward computations while (2.48) corresponds to the L^1-local central limit theorem for densities since W has bounded probability density and bounded third moment (see [10] [page 193] or Theorem 18 in [11] [Chapter VII] where the boundedness of p_W is required). □

Lemma 3. *The random variables X and V in the proof of Theorem 1 have bounded continuous probability densities. Moreover, they have finite nth moment for all $n \in \mathbb{N}$.*

Proof. Due to Theorem 3 in [12] [Section XV.3] in order to prove that X and V have bounded continuous probability densities it is enough that the associated Fourier transforms are in

$L^1(\mathbb{R}, dx)$. Since $X = (\tau-\sigma)+\sigma+\sigma'$ and $\ell_+ = (\tau-\sigma)+\sigma'$ where the random variables $\tau-\sigma, \sigma$ and σ' are independent, it is enough to prove that the Fourier transform of $\tau - \sigma$ is integrable. To this end, we observe that due to Lemma 1 the expectation $\mathbf{E}_0^\mu\left(\mathrm{e}^{-\alpha(\tau-\sigma)}\right)$ is finite for $\alpha > -\mu^2/2$. This implies that the complex Laplace transform $\mathbb{C} \ni \alpha \to \mathbf{E}_0^\mu\left(\mathrm{e}^{-\alpha(\tau-\sigma)}\right) \in \mathbb{C}$ is well defined (i.e. the integrand is integrable) and analytic on the complex halfplane $\Re(\alpha) > -\mu^2/2$. Indeed, integrability is stated in Section 2.2 of [13] and analyticity is stated in Satz 1 (Proposition 1) in Section 3.2 of [13]. We point out that in [13] the author considers the complex Laplace transform of functions, but all arguments and results can be easily extended to the complex Laplace transform of probability measures.

In particular, we get that the Fourier transform $\tau - \sigma$ is given by

$$\mathbf{E}_0^\mu\left(\exp\left(-\mathrm{i}a(\tau-\sigma)\right)\right) = \frac{\sqrt{2a\mathrm{i}+\mu^2}}{\mu}\frac{\sinh(\mu h)}{\sinh\left(\sqrt{2a\mathrm{i}+\mu^2 h}\right)}, \quad a \in \mathbb{R},$$

where the square-root is defined by analytic extension as $\sqrt{r\mathrm{e}^{\mathrm{i}\theta}} = \sqrt{r}\mathrm{e}^{\mathrm{i}\theta/2}$ on the simply connected set $\{r\mathrm{e}^{\mathrm{i}\theta} : r \geq 0, \theta \in (-\pi, \pi)\}$. From the above expression, one easily derives the integrability of the above Fourier transform.

Since the Laplace transforms (2.5) and (2.7) are analytic in the origin, it follows from [12] [Section XIII.2] that σ, $\tau - \sigma$ and σ' have finite nth moments for all $n \in \mathbb{N}$. Hence, the same holds for X and V. $\square$

3. Comparison with the RG-approach

In this section we give some comments on the results concerning the h-extrema of drifted Brownian motion obtained in [8] via the non-rigorous Real Space Renormalization Group (RSRG) method and applied for the analysis of 1D random walks in random environments. We present the results obtained in [8] in the formalism of Sinai's random walk, keeping the discussion at a non-rigorous level.

Start with a sequence of i.i.d. random variables $\{\omega_x\}_{x\in\mathbb{Z}}$ such that $\omega_x \in (0, 1)$ and

$$A := \mathbf{E}\left[\log\frac{1-\omega_0}{\omega_0}\right] \in \mathbb{R}\setminus\{0\}, \qquad \mathrm{Var}\left[\log\frac{1-\omega_0}{\omega_0}\right] =: 2\sigma \in (0,\infty).$$

Defining $\delta := A/(2\sigma)$, the random variable $\log\omega_x/(1-\omega_x)$ corresponds then to the random variable f in (27) of [8]. Without loss of generality we set $\sigma = 1$ as in [8], thus implying that $A = 2\delta$.

The associated Sinai's random walk is the nearest neighbor random walk on $\mathbb{Z}$ where ω_x is the probability to jump from x to $x + 1$ and $1 - \omega_{x-1}$ is the probability to jump from x to $x - 1$. Consider the function $V : \mathbb{R} \to \mathbb{R}$ defined on $\mathbb{Z}$ as

$$V(x) = \begin{cases} \sum_{i=0}^{x-1}\log\frac{1-\omega_x}{\omega_x}, & \text{if } x \geq 1, x \in \mathbb{Z}, \\ 0 & \text{if } x = 0, \\ -\sum_{i=x}^{-1}\log\frac{1-\omega_x}{\omega_x} & \text{if } x < 0, x \in \mathbb{Z}, \end{cases}$$

and extended to all $\mathbb{R}$ by linear interpolation. Usually, the above Sinai's random walk is well described by a diffusion in the potential V. In [8] the authors obtain results on the statistics of the Γ-extrema of V taking the limits $\Gamma \uparrow \infty$, $\delta \downarrow 0$ with $\Gamma\delta$ fixed. In what follows, we show the link between their results and our analysis of the statistics of h-extrema of drifted Brownian motion.

By the Central Limit Theorem applied to $V(x) - 2\delta x$, one concludes that for Γ large

$$\frac{V(x\Gamma^2)}{\sqrt{2}\Gamma} \sim B^*_x + \sqrt{2}\delta\Gamma x, \quad x \in \mathbb{R},$$

where B^* is the standard two-sided Brownian motion (i.e. B^* has law $\mathbf{P}_0$). If we set $\mu := -\sqrt{2}\delta\Gamma$ and consider the limits $\Gamma \uparrow \infty$ and $\delta \downarrow 0$ with μ fixed we get that the rescaled potential V is well approximated by the Brownian motion B with law $\mathbf{P}_0^\mu$. In particular, for Γ large one expects that the ordered family $\{(x_k, V(x_k))\}_{k\in\mathbb{Z}}$ of Γ-extrema of V is well approximated by family $\left\{(\Gamma^2\mathfrak{m}_k, \sqrt{2}\Gamma B_{\mathfrak{m}_k})\right\}_{k\in\mathbb{Z}}$ where $\{\mathfrak{m}_k\}_{k\in\mathbb{Z}}$ is the ordered sequence of points of h-extrema of B with $h = 1/\sqrt{2}$ (we follow the convention that $x_0 \le 0 < x_1$). Setting as in [8] $\zeta := |V(x_{k+1}) - V(x_k)| - \Gamma$ and $l := x_{k+1} - x_k$, we get

$$\zeta/\Gamma \sim \sqrt{2}\left|B_{\mathfrak{m}_{k+1}} - B_{\mathfrak{m}_k}\right| - 1, \qquad l/\Gamma^2 \sim \mathfrak{m}_{k+1} - \mathfrak{m}_k. \tag{3.1}$$

In [8] the authors write $P^+(\zeta, l)\mathrm{d}\zeta\mathrm{d}l$, $P^-(\zeta, l)\mathrm{d}\zeta\mathrm{d}l$ for the joint probability density of the random variables ζ, l if x_k is a point of Γ-minimum or a point of Γ-maximum respectively, they set $P^\pm_\Gamma(\zeta, p) := \int_0^\infty \mathrm{e}^{-lp} P_\Gamma(\zeta, l)\mathrm{d}l$ and derive via the RSRG method the limiting form of $P^\pm_\Gamma(\zeta, p)$ (note that in [8] the authors erroneously do not distinguish the Γ-slope covering the origin from the other Γ-slopes, but in order to have a correct result one must take $k \neq 0$).

It is simple to check that all the computations obtained in [8] [Section II.C.2] equal the results obtained in Proposition 1 by approximating the random variables ζ, l with distribution $P^\pm(\zeta, l)\mathrm{d}\zeta\mathrm{d}l$ by means of the random variables $\sqrt{2}\Gamma\zeta_\pm$, $\Gamma^2\ell_\pm$ of Proposition 1, where $\mu = -\sqrt{2}\delta\Gamma$, $h = 1/\sqrt{2}$. This confirms (3.1).

4. Proof of Lemma 1 via fluctuation theory

Given a path $f \in C([0,\infty), \mathbb{R})$, define the hitting time of f at x as

$$T_x(f) = \inf\{t > 0 : f_t = x\}, \quad x \in \mathbb{R}. \tag{4.1}$$

Consider the process $\{B_t, t \ge 0\}$ carrying the law $\mathbf{P}_0^\mu$ and define (see Fig. 4)

$$\begin{cases} b_t = \min\{B_s : 0 \le s \le t\} \\ L_t = -b_t, \\ Y_t = B_t - b_t. \end{cases} \tag{4.2}$$

The process Y is the so-called one-sided drifted BM reflected at its last infimum. It has the following properties:

Lemma 4 (*[14] [Lemma VI.55.1]*). *The process $Y = \{Y_t, t \ge 0\}$ is a diffusion and $L = \{L_t, t \ge 0\}$ is a local time of Y at 0. The transition density function of the process Y stopped at 0, i.e. $\left\{Y_{t\wedge T_0}, t \ge 0\right\}$, is*

$$\bar{p}_t(x, y) = (2\pi t)^{-1/2}\mathrm{e}^{\mu(x-y)-\frac{1}{2}\mu^2 t}\left[\mathrm{e}^{-(y-x)^2/2t} - \mathrm{e}^{-(y+x)^2/2t}\right], \quad x, y > 0. \tag{4.3}$$

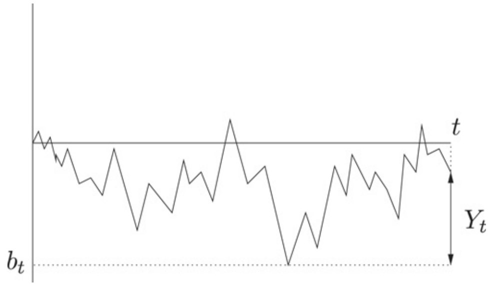

Fig. 4. The process Y_t.

The entrance law n_t, $t > 0$, associated with the excursions of Y from 0 w.r.t. the time L is given by $n_t(\mathrm{d}y) = n_t(y)\mathrm{d}y$, where:

$$n_t(y) = 2y(2\pi t^3)^{-1/2} \exp\left[-(y+\mu t)^2/2t\right], \quad y > 0. \tag{4.4}$$

Let us comment on the above result and fix some notation (for a general treatment of excursion theory see for example [14] [Chapter VI] and [4] [Chapter IV]).

We denote by U the space of excursions from 0, i.e. continuous functions $f : [0,\infty) \to \mathbb{R}$ satisfying the coffin condition

$$f(t) = f(H) = 0, \quad \forall t \geq H,$$

where H is the life time of f, namely

$$H = H(f) = T_0(f) \in [0,\infty]. \tag{4.5}$$

The excursion space U is endowed with the smallest σ-algebra which makes each evaluation map $f \to f(t)$ measurable. One can prove that this σ-algebra coincides with the Borel σ-algebra of the space U endowed with the Skohorod metric.

Write γ_t for the right continuous inverse of L_t, namely

$$\gamma_t = \inf\{u > 0 : L_u > t\} = \inf\{u > 0 : \min_{0\leq s\leq u} B_s < -t\},$$

and define the excursion $e_t \in U$, $t > 0$, as

$$e_t(s) = \begin{cases} Y(\gamma_{t-} + s) & \text{if } 0 \leq s \leq \gamma_t - \gamma_{t-}, \\ 0 & \text{if } s \geq \gamma_t - \gamma_{t-}. \end{cases}$$

Then the random point process ν of excursions of Y from 0 is defined as

$$\nu = \{(t, e_t) : t > 0, \gamma_t \neq \gamma_{t-}\}.$$

In what follows, we will often identify the random discrete set ν with the random measure $\sum_{t>0:\gamma_t\neq\gamma_{t-}} \delta_{(t,e_t)}$ on $(0,\infty) \times U$. Decomposing U as $U = U_\infty \cup U_0$, where

$$U_\infty = \{f \in U : H(f) = \infty\}, \qquad U_0 = \{f \in U : H(f) < \infty\},$$

the Itô Theorem [14] [Theorem VI.47.6] states that there exists a σ-finite measure n on U (called the Itô measure) with $n(U_\infty) < \infty$ such that, if ν' is a Poisson point process on $(0,\infty) \times U$ with intensity measure $\mathrm{d}t \times n$ and if

$$\zeta = \inf\left\{t > 0 : \nu'((0,t] \times U_\infty) > 0\right\}, \tag{4.6}$$

then the point process ν under $\mathbf{P}_0^\mu$ has the same law as $\nu'|_{(0,\zeta]\times U}$:

$$\nu \sim \nu'|_{(0,\zeta]\times U}. \tag{4.7}$$

Here and in what follows, given a measurable space X with measure m and a measurable subset $A \subset X$ we denote by $m|_A$ the measure on X such that $m|_A(B) = m(A \cap B)$ for all $B \subset X$ measurable.

Given $t > 0$ the entrance law $n_t(\mathrm{d}y)$, with support in $(0, \infty)$, is defined as

$$n_t(\mathrm{d}y) = n\left(\{f : H(f) > t,\ f_t \in \mathrm{d}y\}\right). \tag{4.8}$$

Since the process Y (starting at 0) defined via (4.2) is Markov and visits each $y \geq 0$ a.s., the definition of the process Y starting at y is obvious. Due to Lemma 4, the process Y starting at y and stopped at 0 is a strong Markov process with transition probability $\bar{p}_t(\cdot,\cdot)$. In what follows we denote its law by Q_y.

Then, given $t > 0$, measurable subsets $\mathcal{A}, \mathcal{C} \subset U$ with $\mathcal{A} \in \sigma(f_s, 0 \leq s \leq t)$, it holds

$$n\left(f : f \in \mathcal{A}, H(f) > t, \theta_t f \in \mathcal{C}\right) = \int_0^\infty n\left(f \in \mathcal{A}, H(f) > t, f_t \in \mathrm{d}y\right) Q_y(\mathcal{C}), \tag{4.9}$$

where $(\theta_t f)_s = f_{t+s}$. In particular, due to (4.9) the transition density functions (4.3) and the entrance laws (4.4) determine univocally the Itô measure n.

In order to obtain more information on the Itô measure n of the point process of excursions of Y from 0, we give an alternative probabilistic interpretation of the transition density $\bar{p}(x, y)$. To this end, recall that the Girsanov formula implies that

$$\mathbf{E}_x^\mu(g) = \mathbf{E}_x(gZ_t), \qquad Z_t = \exp\left\{-\mu(B_t - x) - \mu^2 t/2\right\}, \tag{4.10}$$

for each $\mathcal{F}_t$-measurable function g, where $\mathcal{F}_t = \sigma\left(B_s : 0 \leq s \leq t\right)$. Due to (4.10), we get for all $x, y, z, s, t > 0$,

$$\begin{aligned} &\mathbf{P}_z^\mu\left(B_{s+t} \in \mathrm{d}y,\ T_0(\theta_s B) > t \mid B_s = x\right) = \mathbf{P}_x^\mu\left(B_t \in \mathrm{d}y,\ T_0 > t\right) \\ &\quad = \mathrm{e}^{-\mu(y-x)-\mu^2 t/2}\mathbf{P}_x\left(B_t \in \mathrm{d}y T_0 > t\right) = \bar{p}_t^\mu(x, y). \end{aligned} \tag{4.11}$$

In fact, the second identity follows from (4.10) while the last identity follows from (4.3) by computing $\mathbf{P}_x\left(B_t \in \mathrm{d}y, T_0 > t\right)$ via a reflection argument. Hence, given $z > 0$, $\bar{p}_t(\cdot,\cdot)$ is the transition density function of the process $\left(B_{t\wedge T_0}, t \geq 0\right)$ under $\mathbf{P}_z^\mu$, whose law equals Q_z. In particular, (4.9) can be reformulated as

$$\begin{aligned} &n\left(f : f \in \mathcal{A}, H(f) > t, \theta_t f \in \mathcal{C}\right) \\ &\quad = \int_0^\infty n\left(f \in \mathcal{A}, H(f) > t, f_t \in \mathrm{d}y\right) \mathbf{P}_y^\mu(B_{\cdot\wedge T_0} \in \mathcal{C}). \end{aligned} \tag{4.12}$$

The above identity will be frequently used in what follows.

We point out that, as stated in Theorem 1 of [4] [Section VII.1], the content of Lemma 4 is valid in more generality for spectrally positive Lévy processes s.t. the origin is a regular point, i.e. real valued processes starting at the origin with stationary independent increments, with no negative jumps and returning to the origin at arbitrarily small times. Moreover, defining $\tilde{b}_t := 0 \wedge \inf\{B_s : 0 \leq s \leq t\}$, it is simple to check that the process Y starting at $x > 0$ has the same law of the process $\left(\tilde{Y}_t := B_t - \tilde{b}_t, t \geq 0\right)$, where B is chosen with law $\mathbf{P}_x^\mu$. This implies that $\tilde{Y}_t = B_t$ if $t < T_0(B)$, hence one gets again that $\bar{p}^{\mu}_t(x, y) = \mathbf{P}^{\mu}_x(B_t \in \mathrm{d}y, T_0 > t)$ as in (4.11).

In order to state our results it is useful to fix some further notation. Given $h > 0$ we denote by $U^{h,+}$ the family of excursions with height at least h and by $U^{h,-}$ the family of excursions with height less than h, namely

$$U^{h,+} = \left\{ f \in U : \sup_{s\geq 0} f_s \geq h \right\}, \qquad U^{h,-} = \left\{ f \in U : \sup_{s\geq 0} f_s < h \right\}.$$

One of the main technical tools in order to extend the proof in [1] to the drifted case is the following lemma, whose proof is postponed to Section 6.

Lemma 5. *If $\hat{\alpha} > 0$, $\mu \neq 0$, then*

$$n\left(U^{h,+}\right) = \frac{\mu \mathrm{e}^{-\mu h}}{\sinh(\mu h)}, \tag{4.13}$$

$$n\left(U^{h,-} \cap U_\infty\right) = 0, \tag{4.14}$$

$$\int_{U^{h,-}} \left(1 - \mathrm{e}^{-\alpha H(f)}\right) n(\mathrm{d}f) = \sqrt{2\hat{\alpha}} \coth\left(\sqrt{2\hat{\alpha}} h\right) - \mu \coth(\mu h), \tag{4.15}$$

$$n\left(U^{h,+}\right)^{-1} \int_{U^{h,+}} \mathrm{e}^{-\alpha T_h} n(\mathrm{d}f) = \frac{\sqrt{2\hat{\alpha}}}{\mu} \frac{\sinh(\mu h)}{\sinh\left(\sqrt{2\hat{\alpha}} h\right)}. \tag{4.16}$$

Finally, we can prove Lemma 1:

Proof of Lemma 1. One can recover the Brownian motion from the point process ν of excursions of Y from 0 by the formula

$$B_t = -a + f(t - S), \quad \text{for } t \in [S, S + H(f)], \tag{4.17}$$

which is valid for each couple $(a, f) \in \nu$ by setting

$$S = \int_{(0,a)\times U} H(f')\nu(\mathrm{d}a', \mathrm{d}f').$$

It is convenient to associate to $U^{h,+}$, $U^{h,-}$ the measures $\nu^* = \nu|_{[0,\infty)\times U^{h,+}}$, $\nu_* = \nu|_{[0,\infty)\times U^{h,-}}$, $n^* = n|_{U^{h,+}}$ and $n_* = n|_{U^{h,-}}$. Moreover, we set

$$a^* = \inf\left\{a > 0 : \exists f \in U^{h,+} \text{ with } (a, f) \in \nu\right\}.$$

If a^* is finite, let f^* be the only excursion such that $(a^*, f^*) \in \nu$. Due to (4.6), (4.7) and (4.14)

$$\mathbf{P}^{\mu}_0(a^* > a) = \mathbf{P}^{\mu}_0\left(\nu^*\left((0, a] \times U\right) = 0\right) = \exp\left\{-a\, n^*(U)\right\},$$

therefore a^* is an exponential variable with parameter $n^*(U)$ (in particular, a^* is finite a.s.). Due to representation (4.17), $\beta = -a^*$. Together with (4.13) this implies that $-\beta$ is an exponential variable with mean (2.4). Moreover, (4.17) implies that

$$\sigma = \int_{(0,a^*)\times U} H(f')\nu(\mathrm{d}a', \mathrm{d}f') = \int_{(0,a^*)\times U} H(f')\nu_*(\mathrm{d}a', \mathrm{d}f'). \tag{4.18}$$

Due to the above expression and representation (4.17), the trajectory $(B_t, 0 \le t \le \sigma)$ depends only on ν_* and a^*, while the trajectory $(B_{\sigma+t} - \beta, 0 \le t \le \tau - \sigma)$ coincides with the excursion f^* stopped when it reaches level h. Since ν_* and a^* are independent from f^* we get the independence of the trajectories.

In order to prove (2.5) we observe that $-\beta = x$ means that $a^* = x$. Therefore, conditioning to $-\beta = x$, it holds $\sigma = \int_{(0,x)\times U} H(f')\nu_*(\mathrm{d}a', \mathrm{d}f')$, thus implying that

$$\mathbf{E}_0^\mu \left[\exp(-\alpha\sigma) \mid \beta = -x\right] = \mathbf{E}_0^\mu \left(\exp\left\{-\alpha \int_{(0,x)\times U} H(f')\nu_*(\mathrm{d}a', \mathrm{d}f')\right\}\right). \tag{4.19}$$

Note that, in order to derive the above identity, we have used that ν is the superposition of the independent point processes ν^* and ν_*.

We claim that

$$\begin{aligned}\mathbf{E}_0^\mu &\left(\exp\left\{-\alpha \int_{(0,x)\times U} H(f')\nu_*(da', \mathrm{d}f')\right\}\right) \\ &= \exp\left\{-x \int_U \left(1 - \mathrm{e}^{-\alpha H(f)}\right) n_*(\mathrm{d}f)\right\}.\end{aligned} \tag{4.20}$$

In order to prove this claim we note that, due to the Itô Theorem and (4.14), the point process ν_* has the same distribution of the Poisson point process on $(0,\infty) \times U$ with intensity $\mathrm{d}t \times n_*$. Hence, for $\alpha \ge 0$ the above identity follows directly from the exponential formula for Poisson point processes [4] [Section O.5]. Suppose now that $\alpha < 0$ and $\hat{\alpha} \ge 0$. Given $m > 0$ and $f \in U$ we define $H_m(f)$ as $-\infty$ if $H(f) \le m$ and as $H(f)$ if $H(f) > m$. Due to (4.8) and (4.12), we get the bound

$$\int \mathrm{e}^{-\alpha H_m(f)} n_*(\mathrm{d}f) \le \mathrm{e}^{-\alpha m} \int_0^h n_m(\mathrm{d}y) \mathbf{E}_y^\mu \left(\mathrm{e}^{-\alpha T_0} \mathbb{I}_{T_0 < T_h}\right)$$

where the r.h.s. is finite due to the form of n_m and identity (6.2). This allows us to conclude that the integral $\int \mathrm{e}^{-\alpha H_m(f)} n_*(\mathrm{d}f)$ is finite, and therefore the same holds for the smaller integral $\int \left|1 - \mathrm{e}^{-\alpha H_m(f)}\right| n_*(\mathrm{d}f)$. This last property allows us to apply again the exponential formula for Poisson point processes and to deduce that

$$\mathbf{E}_0^\mu \left(\exp\left\{-\alpha \int_{(0,x)\times U} H_m(f')\nu_*(da', \mathrm{d}f')\right\}\right) = \exp\left\{-x \int_U \left(1 - \mathrm{e}^{-\alpha H_m(f)}\right) n_*(\mathrm{d}f)\right\}.$$

Taking the limit $m \downarrow 0$ and applying the Monotone Convergence Theorem we derive (4.20) from the above identity. Hence, (2.5) follows from (4.15), (4.19) and (4.20), while trivially (2.7) follows from (2.5).

Finally, in order to prove (2.3), we observe that $\tau - \sigma = T_h(f^*)$. Since the path f^* has law $n^*/n^*(U)$, $\mathbf{E}_0^\mu(\exp(-\alpha(\tau - \sigma)))$ equals $\int_U n^*(\mathrm{d}f) \exp\left(-\alpha T_h(f)\right)/n^*(U)$ and the thesis follows from (4.16). $\square$

Remark 2. As already remarked, the analogue of Lemma 1 (restricted to $\alpha > 0$) has been proved for more general spectrally one-sided Lévy processes [5,6] and [7, Proposition 1], by means of more sophisticated arguments always based on fluctuation theory, excursion theory and the analysis of the hitting times of the process. We have given a self-contained and direct proof based on simple computations, which will be useful also for the proof of Theorem 2, but one can derive Lemma 1 from the cited references as follows. The Laplace exponent of the drifted BM with law $\mathbf{P}_0^\mu$ is given by $\psi(\lambda) = \frac{1}{2}\lambda^2 - \lambda\mu$, i.e. $\mathbf{E}_0^\mu\left(\mathrm{e}^{\lambda B_t}\right) = \mathrm{e}^{t\psi(\lambda)}$ for $\lambda \in \mathbb{R}$. Given $\alpha > 0$ we define the function $W^{(\alpha)}(x)$ as

$$W^{(\alpha)}(x) = \mathrm{e}^{\mu x}\left(\mathrm{e}^{x\sqrt{2\hat{\alpha}}} - \mathrm{e}^{-x\sqrt{2\hat{\alpha}}}\right)/\sqrt{2\hat{\alpha}}.$$

Then it is simple to check that

$$\int_0^\infty \mathrm{e}^{-\lambda x} W^{(\alpha)}(x)dx = \frac{1}{\psi(\lambda)-\alpha}, \quad \forall \lambda \geq \Phi(\alpha),$$

where the value $\Phi(\alpha)$ is defined as the largest root of $\psi(\lambda) = \alpha$, i.e. $\Phi(\alpha) = \mu + \sqrt{2\hat{\alpha}}$. The function $W^{(\alpha)}$ is related to the exit of the BM from a given interval. More precisely, due to (6.3), it holds

$$\mathbf{E}_0^\mu(\mathrm{e}^{-\alpha T_y} T_y < T_{-x}) = W^{(\alpha)}(x)/W^{(\alpha)}(x+y), \quad \forall x, y > 0. \tag{4.21}$$

To the function $W^{(\alpha)}$ one associates the function $Z^{(\alpha)}$ given by

$$Z^{(\alpha)}(x) = 1 + \alpha \int_0^x W^{(\alpha)}(z)\mathrm{d}z = \frac{\alpha \mathrm{e}^{\mu x}}{\sqrt{2\hat{\alpha}}}\left(\frac{\mathrm{e}^{\sqrt{2\hat{\alpha}}x}}{\mu+\sqrt{2\hat{\alpha}}} - \frac{\mathrm{e}^{-\sqrt{2\hat{\alpha}}x}}{\mu-\sqrt{2\hat{\alpha}}}\right).$$

Knowing the values of $W^{(\alpha)}$ and $Z^{(\alpha)}$ one can compute the expressions in Lemma 1 for $\alpha > 0$ by applying for example, Proposition 1 in [7].

5. The behavior of the drifted Brownian motion near an *h*-extremum

In this section we characterize the behavior of an h-slope not covering the origin, near its extremes. To this end, we recall the definition of the drifted Brownian motion Doob-conditioned to hit $+\infty$ before 0, referring to [4] [Section VII.2] and references therein for a more detailed discussion. First, we write $W(x)$ for the function

$$W(x) := W^{(0)}(x) = \frac{\mathrm{e}^{2x\mu}-1}{\mu}$$

($W^{(\alpha)}$ has been defined in Remark 2). Defining $\Phi(0)$ as the largest zero of $\psi(\lambda) := \lambda^2/2 - \lambda\mu$, i.e. $\Phi(0) := 0 \vee (2\mu)$, the function W is a positive increasing function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda x} W(x)\mathrm{d}x = \frac{1}{\psi(\lambda)}, \quad \forall \lambda > \Phi(0),$$

satisfying the identity (see (4.21))

$$\mathbf{P}_0^\mu(T_y < T_{-x}) = W(x)/W(x+y), \quad \forall x, y > 0. \tag{5.1}$$

Due to the above considerations, the function W is the so-called *scale function* of the drifted Brownian motion with law $\mathbf{P}_0^\mu$.

For each $x > 0$ consider the new probability measure $\mathbf{P}_x^{\mu,\uparrow}$ on the path space $C([0,\infty),\mathbb{R})$ characterized by the identity

$$\mathbf{P}_x^{\mu,\uparrow}(\Lambda) = \frac{1}{W(x)}\mathbf{E}_x^\mu\left(W(X_t), \Lambda, t < T_0\right), \quad \Lambda \in \mathcal{F}_t, \tag{5.2}$$

where $(X_t, t \geq 0)$ denotes a generic element of the path space $C([0,\infty),\mathbb{R})$ and $\mathcal{F}_t := \sigma\{X_s : 0 \leq s \leq t\}$. As discussed in [4] [Section VII.3], the above probability measure is well defined, the weak limit $\mathbf{P}_0^{\mu,\uparrow} := \lim_{x\downarrow 0} \mathbf{P}_x^{\mu,\uparrow}$ exists and the process $\left(\mathbf{P}_x^{\mu,\uparrow}, x \geq 0\right)$ is a Feller process, hence strong Markov (we point out that in [4] [Section VII.3] the above results are proven in the Skohorod path space $D([0,\infty),\mathbb{R})$, but one can adapt the proofs to $C([0,\infty),\mathbb{R})$). As explained in [4] [Section VII.3], this process can be thought of as the Brownian motion with drift $-\mu$ Doob-conditioned to hit $+\infty$ before 0. In the case of positive drift, i.e. $\mu < 0$, this can be realized very easily by observing that due to (5.1)

$$\begin{aligned}\mathbf{P}_x^{\mu}(T_0=\infty) &= \mathbf{P}_0^{\mu}(T_{-x}=\infty) = \lim_{y\uparrow\infty} \mathbf{P}_0^{\mu}(T_y < T_{-x}) \\ &= \lim_{y\uparrow\infty} \frac{W(x)}{W(x+y)} = 1-\mathrm{e}^{2x\mu} = -\mu W(x), \quad \forall x>0,\end{aligned}$$

and that this identity together with the Markov property implies that $\mathbf{P}_x^{\mu}(\Lambda|T_0=\infty)$ equals $\mathbf{P}_x^{\mu,\uparrow}(\Lambda)$ for all $x>0$, $\Lambda\in\mathcal{F}_t$. For negative drift the event $\{T_0=\infty\}$ has zero probability, and a more subtle discussion is necessary.

Lemma 6. *The process $\mathbf{P}_0^{\mu,\uparrow}$ is a diffusion characterized by the SDE*

$$\mathrm{d}X_t = \mathrm{d}B_t + \mu \coth(\mu X_t)\,\mathrm{d}t, \qquad X_0 = 0, \tag{5.3}$$

where B_t is the standard Brownian motion.

Proof. As already discussed, the above process has continuous paths and it is strong Markov, i.e. it is a diffusion.

Due to (5.2) and (4.11), given $x, y > 0$,

$$\begin{aligned} q_t(x,y) := \mathbf{P}_x^{\mu,\uparrow}(X_t\in \mathrm{d}y) &= \frac{W(y)}{W(x)} P_x^{\mu}(X_t\in \mathrm{d}y, t<T_0) = \frac{W(y)\bar{p}_t^{\mu}(x,y)}{W(x)} \\ &= \frac{\sinh(\mu y)}{\sinh(\mu x)} \frac{\mathrm{e}^{-\frac{1}{2}\mu^2 t}}{\sqrt{2\pi t}} \left[\mathrm{e}^{-(y-x)^2/2t} - \mathrm{e}^{-(y+x)^2/2t}\right]. \end{aligned} \tag{5.4}$$

From the above expression, by direct computations one derives that

$$\frac{\partial}{\partial t} q_t(x,y) = -\frac{\partial}{\partial y}\left(\mu\coth(\mu y) q_t(x,y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2} q_t(x,y). \tag{5.5}$$

Hence, the generator of the process is given by

$$\mathcal{L}f(y) = \mu\coth(\mu y)\frac{\mathrm{d}}{\mathrm{d}y} f(y) + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} f(y)$$

and this implies the SDE (5.3). $\square$

Let us consider now the Brownian motion with drift $-\mu$ Doob-conditioned to hit h before 0 and killed when it reaches h. In order to make precise its meaning when the Brownian motion starts at the origin, given $0<x<h$, we define $\mathbf{P}_{x,h}^{\uparrow,\mu}$ as the conditioned law on $C([0,\infty),\mathbb{R})$ such that $\mathbf{P}_{x,h}^{\mu,\uparrow}(\Lambda) = \mathbf{P}_x^{\mu}(\Lambda \mid T_h < T_0)$ for all $\Lambda \in \cup_{s\geq 0}\mathcal{F}_s$. Note that the above definition is well posed since by (5.1), $\mathbf{P}_x^{\mu}(T_h<T_0) = W(x)/W(h) > 0$.

Lemma 7. *Given $0<x<h$, let $Q_{x,h}^{\mu,\uparrow}$ and $R_{x,h}^{\mu,\uparrow}$ be the laws of the path $(X_t : 0 \leq t \leq T_h)$ killed when level h is reached, where X is chosen with laws $\mathbf{P}_{x,h}^{\mu,\uparrow}$ and $\mathbf{P}_x^{\mu,\uparrow}$ respectively. Then $Q_{x,h}^{\mu,\uparrow} = R_x^{\mu,\uparrow}$ and the weak limit $Q_{0,h}^{\mu,\uparrow} := \lim_{x\downarrow 0} Q_{x,h}^{\mu,\uparrow}$ exists and equals $R_0^{\mu,\uparrow}$.*

Proof. Given $0 < x_1, x_2, \ldots, x_n < h$ and times $t_1 < t_2 < \cdots < t_n$, we denote by $\mathcal{A}$ the event $\mathcal{A} := \left\{X_{t_i} \in dx_i \ \forall i : 1 \le i \le n, t_n < T_h\right\}$. Then, by the definition of $\mathbf{P}^{\mu,\uparrow}_{x,h}$ and the Markov property of the Brownian motion, for each $0 < x < h$ we get that

$$\mathbf{P}^{\mu,\uparrow}_{x,h}(\mathcal{A}) = \mathbf{P}^{\mu}_{x}(\mathcal{A}, t_n < T_0)\mathbf{P}^{\mu}_{x_n}(T_h < T_0)/\mathbf{P}^{\mu}_{x}(T_h < T_0) = \mathbf{P}^{\mu}_{x}(\mathcal{A}, t_n < T_0)W(x_n)/W(x).$$

Due to (5.2), the last expression in the r.h.s. equals the probability $\mathbf{P}^{\mu,\uparrow}_{x}(\mathcal{A})$, hence we can conclude that $\mathbf{P}^{\mu,\uparrow}_{x,h}(\mathcal{A}) = \mathbf{P}^{\mu,\uparrow}_{x}(\mathcal{A})$. Hence $Q^{\mu,\uparrow}_{x,h} = R^{\mu,\uparrow}_{x,h}$ for $0 < x < h$. The last statement concerning $Q^{\mu,\uparrow}_{0,h}$ follows from the fact that the weak limit $\lim_{x\downarrow 0}\mathbf{P}^{\mu,\uparrow}_{x}$ exists and equals $\mathbf{P}^{\mu,\uparrow}_{0}$. $\square$

Due to the first part of the above lemma, we can think of $\left(\mathbf{P}^{\mu,\uparrow}_{x,h}, 0 \le x \le h\right)$ as the Brownian motion with drift $-\mu$ Doob-conditioned to hit h before 0 and killed when it hits h.

Now we have all the tools in order to describe the behavior of the h-slopes not covering the origin, near their extremes. In order to simplify the notation, in what follows we denote by $B^{(\mu)}$ the two-sided Brownian motion with drift $-\mu$, starting at the origin. Moreover, given $r \in \mathbb{R}$, we define $T^{(h,-)}_r$ and $T^{(h,+)}_r$ as

$$T^{(h,\pm)}_r = \inf\left\{s > 0 : \left|B^{(\mu)}_{r\pm s} - B^{(\mu)}_r\right| = h\right\}.$$

Theorem 2. *Let $\mu \neq 0$ and let $m < m'$ be consecutive points of h-extrema for the drifted Brownian motion $B^{(\mu)}$, both non-negative or both non-positive.*

If m is a point of h-minimum and m' is a point of h-maximum, then the processes

$$\left\{B^{(\mu)}_{m+t} - B^{(\mu)}_{m}, 0 \le t \le T^{(h,+)}_m\right\}, \tag{5.6}$$

$$\left\{B^{(\mu)}_{m'} - B^{(\mu)}_{m'-t}, 0 \le t \le T^{(h,-)}_{m'}\right\}, \tag{5.7}$$

have the same law of the Brownian motion starting at the origin, with drift $-\mu$, Doob-conditioned to reach $+\infty$ before 0 and killed when it hits h. Moreover, they have the same law of the Brownian motion starting at the origin, with drift $-\mu$, Doob-conditioned to reach h before 0 and killed when it hits h. In particular, they satisfy the SDE (5.3) up to the killing time.

If m is a point of h-maximum and m' is a point of h-minimum, then the processes

$$\left\{B^{(\mu)}_{m} - B^{(\mu)}_{m+t}, 0 \le t \le T^{(h,+)}_m\right\}, \tag{5.8}$$

$$\left\{B^{(\mu)}_{m'-t} - B^{(\mu)}_{m'}, 0 \le t \le T^{(h,-)}_{m'}\right\}, \tag{5.9}$$

have the same law of the Brownian motion starting at the origin, with drift μ, Doob-conditioned to reach $+\infty$ before 0 and killed when it hits h. Moreover, they have the same law of the Brownian motion starting at the origin, with drift μ, Doob-conditioned to reach h before 0 and killed when it hits h. In particular, they satisfy the SDE (5.3) with μ replaced by $-\mu$, up to the killing time.

Proof. The second part of the theorem follows from the first part by taking the reflection w.r.t. the coordinate axis. As follows from the proof of Lemma 1 in Section 4, the law of process (5.6) coincides with the law of the excursion f killed when it reaches h, where f is chosen with probability measure $n(\cdot|T_h < T_0) = n(\cdot, T_h < T_0)/n(T_h < T_0)$. Due to Proposition 15 in [4] [Section VII.3], there exists a positive constant c such that

$$n(\Lambda, t < T_0) = c\,\mathbf{E}_0^{\mu,\uparrow}\left(W(X_t)^{-1}, \Lambda\right), \quad \forall \Lambda \in \mathcal{F}_t, \tag{5.10}$$

$$n(T_h < T_0) = c/W(h) \tag{5.11}$$

(note that (5.11) corresponds to (4.13) with $c = 2$). Given numbers $x_1, \ldots, x_n$ in $(0, h)$ and increasing times $0 < t_1 < \cdots < t_n$, setting $\mathcal{A} := \{f_{t_i} \in dx_i \ \forall i : 1 \le i \le n,\ t_n < T_h\}$, we have

$$n(\mathcal{A}, T_h < T_0) = n(\mathcal{A}, t_n < T_0)\,\mathbf{P}_{x_n}^{\mu}(T_h < T_0) = c\,\mathbf{P}_0^{\mu,\uparrow}(\mathcal{A})/W(h) \tag{5.12}$$

(the first identity follows from the Markov property (4.12), while the latter follows from (5.1) and (5.10)). From (5.11) and (5.12) one derives that $n(\mathcal{A}|T_h < T_0) = \mathbf{P}_0^{\mu,\uparrow}(\mathcal{A})$. This concludes the proof that process (5.6) has the same law of the Brownian motion starting at the origin, with drift $-\mu$, Doob-conditioned to reach $+\infty$ before 0 and killed when it hits h. All the other statements follow from this property, Lemmata 6 and 7 and by reflection arguments. $\square$

6. Proof of Lemma 5

Knowing the Laplace transform of the hitting times of the non-drifted Brownian motion [15] [Chapter 2], the following lemma follows by applying the Girsanov formula (4.10):

Lemma 8. *Let $x < 0 < y$ and $\hat{\alpha} > 0$. If $\{u, v\} = \{x, y\}$, then*

$$\mathbf{E}_0^{\mu}\left(\mathrm{e}^{-\alpha T_u}\mathbb{I}_{T_u < T_v}\right) = \mathrm{e}^{-\mu u}\frac{\sinh\left(|v|\sqrt{2\hat{\alpha}}\right)}{\sinh\left(|u - v|\sqrt{2\hat{\alpha}}\right)}. \tag{6.1}$$

Proof. Take $u = x$, $v = y$ (the other case $u = y$, $v = x$ can be treated similarly). Set $Z_t = \exp\left\{-\mu B_t - \mu^2 t/2\right\}$. We claim that

$$\begin{aligned}
\mathbf{E}_0^{\mu}\left(\mathrm{e}^{-\alpha T_x}\mathbb{I}_{T_x < T_y}\right) &= \lim_{t\uparrow\infty}\mathbf{E}_0^{\mu}\left(\mathrm{e}^{-\alpha T_x}\mathbb{I}_{T_x < T_y}\mathbb{I}_{T_x < t}\right)\\
&= \lim_{t\uparrow\infty}\mathbf{E}_0\left(\mathrm{e}^{-\alpha T_x}\mathbb{I}_{T_x < T_y}\mathbb{I}_{T_x < t} Z_t\right)\\
&= \lim_{t\uparrow\infty}\mathbf{E}_0\left(\mathrm{e}^{-\alpha T_x}\mathbb{I}_{T_x < T_y}\mathbb{I}_{T_x < t}\mathbf{E}_0(Z_t \mid \mathcal{F}_{t\wedge T_x})\right).
\end{aligned}$$

Indeed, the first identity follows from the Monotone Convergence Theorem, the second one from (4.10), and the last one by conditioning on $\mathcal{F}_{t\wedge T_x}$ and observing that $\mathrm{e}^{-\alpha T_x}\mathbb{I}_{T_x < T_y}\mathbb{I}_{T_x < t}$ is $\mathcal{F}_{t\wedge T_x}$ measurable.

Since, under $\mathbf{P}_0$, Z_t is a martingale and $t \wedge T_x$ is a bounded stopping time, the optional sampling theorem implies that $\mathbf{E}\left(Z_t \mid \mathcal{F}_{t\wedge T_x}\right) = Z_{t\wedge T_x}$. For $T_x < t$, $Z_{t\wedge T_x}$ equals $\exp\left(-\mu x - \hat{\alpha} T_x + \alpha T_x\right)$, hence

$$\mathbf{E}_0^{\mu}\left(\mathrm{e}^{-\alpha T_x}\mathbb{I}_{T_x < T_y}\right) = \lim_{t\uparrow\infty}\mathrm{e}^{-\mu x}\mathbf{E}_0\left(\mathrm{e}^{-\hat{\alpha} T_x}\mathbb{I}_{T_x < T_y}\mathbb{I}_{T_x < t}\right) = \mathrm{e}^{-\mu x}\mathbf{E}_0\left(\mathrm{e}^{-\hat{\alpha} T_x}\mathbb{I}_{T_x < T_y}\right).$$

Since $\hat{\alpha} > 0$, the above identity and formula (8.27) in [15] [Chapter 2] allow us to conclude. $\square$

Due to the above lemma, given $0 < y < h$ and $\hat{\alpha} > 0$,

$$\mathbf{E}_y^{\mu}\left(\mathrm{e}^{-\alpha T_0}\mathbb{I}_{T_0<T_h}\right) = \mathrm{e}^{\mu y}\frac{\sinh\left((h-y)\sqrt{2\hat{\alpha}}\right)}{\sinh\left(h\sqrt{2\hat{\alpha}}\right)}, \tag{6.2}$$

$$\mathbf{E}_y^{\mu}\left(\mathrm{e}^{-\alpha T_h}\mathbb{I}_{T_h<T_0}\right) = \mathrm{e}^{\mu(y-h)}\frac{\sinh\left(y\sqrt{2\hat{\alpha}}\right)}{\sinh\left(h\sqrt{2\hat{\alpha}}\right)}, \tag{6.3}$$

$$\mathbf{P}_y^{\mu}\left(T_0 < T_h\right) = \mathrm{e}^{\mu y}\frac{\sinh\left(\mu(h-y)\right)}{\sinh(\mu h)}, \tag{6.4}$$

$$\mathbf{P}_y^{\mu}\left(T_h < T_0\right) = \mathrm{e}^{\mu(y-h)}\frac{\sinh\left(\mu y\right)}{\sinh(\mu h)}. \tag{6.5}$$

By taking the limit $h \to \infty$ in (6.2) we get for all $y > 0$ and $\hat{\alpha} > 0$

$$\mathbf{E}_y^{\mu}\left(\mathrm{e}^{-\alpha T_0}\mathbb{I}_{T_0<\infty}\right) = \mathrm{e}^{\mu y - y\sqrt{2\hat{\alpha}}} = \mathrm{e}^{\mu y-|\mu|y\sqrt{1+\frac{2\alpha}{\mu^2}}}. \tag{6.6}$$

By considering the Taylor expansion around $\alpha = 0$ in the above identity, one can compute the expectation of $T_0^k\,\mathbb{I}_{T_0<\infty}$. In particular, for all $y > 0$ it holds

$$\mathbf{E}_y^{\mu}(T_0\mathbb{I}_{T_0<\infty}) = \mathrm{e}^{y(\mu-|\mu|)}y/|\mu|. \tag{6.7}$$

We collect some identities (obtained by straightforward computations) which will be very useful below. First we observe that given $a, b, w \in \mathbb{R}$ and $t > 0$ it holds

$$\begin{aligned}\int_a^b \frac{2y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y+wt)^2}{2t}}\mathrm{d}y = {} & \frac{2}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{(a+wt)^2}{2t}} - \frac{2}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{(b+wt)^2}{2t}} \\ & - \frac{2w}{\sqrt{2\pi}}\int_{a/\sqrt{t}+w\sqrt{t}}^{b/\sqrt{t}+w\sqrt{t}}\mathrm{e}^{-\frac{z^2}{2}}\mathrm{d}z.\end{aligned} \tag{6.8}$$

In particular, for fixed $a > 0$ and $c, w \in \mathbb{R}$, it holds as $t \downarrow 0$

$$\int_a^{\infty}\frac{2y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y+wt)^2}{2t}}\mathrm{d}y = o(1) \tag{6.9}$$

$$\int_0^a \frac{2y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y+wt)^2}{2t}-cy}\mathrm{d}y = \frac{2}{\sqrt{2\pi t}} - (w+c) + o(1) \quad \text{as } t \downarrow 0. \tag{6.10}$$

As the first application of the above observations and (4.12), we prove the following result:

Lemma 9. *Let $\hat{\alpha} > 0$. Then*

$$\lim_{t\downarrow 0}\int_U \left|1-\mathrm{e}^{-\alpha H(f)}\right|\mathbb{I}_{H(f)\le t}\,n(\mathrm{d}f) = 0. \tag{6.11}$$

Proof. It is enough to prove that $\lim_{t\downarrow 0}\int_U H(f)\mathbb{I}_{H(f)\le t}n(\mathrm{d}f) = 0$. Then, by the Dominated Convergence Theorem, it is enough to show that $\int_U H(f)\mathbb{I}_{H(f)\le 1}n(\mathrm{d}f) < \infty$. Then, by the Monotone Convergence Theorem, we only need to show that $\lim_{\varepsilon\downarrow 0}\int_U H(f)\mathbb{I}_{\varepsilon<H(f)\le 1}n(\mathrm{d}f)$ is finite. Due to (4.12), the previous integral equals $\int n_\varepsilon(\mathrm{d}y)\mathbf{E}_y^{\mu}((T_0+\varepsilon)\mathbb{I}_{T_0\le 1-\varepsilon})$, which can be bounded by

$$\int n_\varepsilon(\mathrm{d}y)\mathbf{E}_y^\mu(T_0\mathbb{I}_{T_0<\infty}) + \varepsilon \int n_\varepsilon(\mathrm{d}y). \tag{6.12}$$

Due to (6.10) and (6.9), $\varepsilon \int n_\varepsilon(\mathrm{d}y)$ is negligible as $\varepsilon \downarrow 0$ while, due to (6.7), the first term in Eq. (6.12) equals $(\varepsilon|\mu|)^{-1}\int_0^\infty \frac{2y^2}{\sqrt{2\pi\varepsilon}}\mathrm{e}^{-\frac{(y+|\mu|\varepsilon)^2}{2\varepsilon}}\mathrm{d}y$, which can be bounded by

$$\frac{1}{\varepsilon|\mu|}\int_0^\infty \frac{2(y+|\mu|\varepsilon)^2}{\sqrt{2\pi\varepsilon}}\mathrm{e}^{-\frac{(y+|\mu|\varepsilon)^2}{2\varepsilon}}\mathrm{d}y = \frac{1}{|\mu|} < \infty. \quad \square$$

Now we have all the technical tools in order to prove Lemma 5.

6.1. *Proof of* (4.13)

Consider the measurable subsets $U_t^{h,+} = \left\{f \in U : \sup_{s\geq t} f_s \geq h,\, H(f) > t\right\}$. Then $U_{t_2}^{h,+} \subset U_{t_1}^{h,+}$ for $t_1 \leq t_2$ and $U^{h,+} = \cup_{t>0} U_t^{h,+}$. This implies that $n\left(U^{h,+}\right) = \lim_{t\downarrow 0} n\left(U_t^{h,+}\right)$. Due to (4.12)

$$n\left(U_t^{h,+}\right) = \int_0^\infty n_t(\mathrm{d}y)\mathbf{P}_y^\mu\left(\sup_{s\geq 0} B_{s\wedge T_0} \geq h\right) = \int_0^\infty n_t(\mathrm{d}y)\mathbf{P}_y^\mu\left(T_h < T_0\right).$$

Hence (see also (6.5)) we get that $n\left(U^{h,+}\right) = \lim_{t\downarrow 0}(I_1(t) + I_2(t))$, where

$$I_1(t) = \int_h^\infty \frac{2y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y+\mu t)^2}{2t}}\mathrm{d}y, \qquad I_2(t) = \int_0^h \frac{2y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y+\mu t)^2}{2t}}\mathrm{e}^{\mu(y-h)}\frac{\sinh(\mu y)}{\sinh(\mu h)}\mathrm{d}y.$$

Due to (6.9), $\lim_{t\downarrow 0} I_1(t) = 0$. Let us write $I_2(t)$ as $(I_3(t) - I_4(t))/(1-\mathrm{e}^{2\mu h})$, where

$$I_3(t) = \int_0^h \frac{2y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y+\mu t)^2}{2t}}\mathrm{d}y,$$
$$I_4(t) = \int_0^h \frac{2y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y+\mu t)^2}{2t}+2\mu y}\mathrm{d}y = \int_0^h \frac{2y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y-\mu t)^2}{2t}}\mathrm{d}y.$$

Due to (6.10), $I_3(t) = \frac{2}{\sqrt{2\pi t}} - \mu + o(1)$ and $I_4(t) = \frac{2}{\sqrt{2\pi t}} + \mu + o(1)$, thus allowing us to conclude. $\square$

6.2. *Proof of* (4.14)

Consider the subsets $U_t \subset U$ defined as $U_t = \left\{f \in U : H(f) = \infty, \sup_{s\geq t} f_s < h\right\}$. Then $U^{h,-} \cap U_\infty \subset U_t$ and it is enough to show that $\lim_{t\downarrow 0} n(U_t) = 0$. Due to (4.12), $n(U_t)$ equals $\int_0^\infty n_t(\mathrm{d}y)\mathbf{P}_y^\mu(T_h = T_0 = \infty)$. If $\mu > 0$ then $\mathbf{P}_y^\mu(T_0 = \infty) = 0$ for all $y > 0$, while if $\mu < 0$ then $\mathbf{P}_y^\mu(T_h = \infty) = 0$ for all $y < h$. Hence $n(U_t) \leq \int_h^\infty n_t(\mathrm{d}y)$. By (6.9) the last member above goes to 0 as $t \downarrow 0$, thus allowing us to conclude. $\square$

6.3. *Proof of* (4.15)

By applying the Dominated Convergence Theorem and the Monotone Convergence Theorem, it is simple to prove that

$$\int_{U^{h,-}}\left(1-\mathrm{e}^{-\alpha H(f)}\right)n(\mathrm{d}f) = \lim_{t\downarrow 0}\int_U \left(1-\mathrm{e}^{-\alpha H(f)}\right)\mathbb{I}_{\{\sup_{[t,\infty)} f<h\}}\mathbb{I}_{H(f)>t}\,n(\mathrm{d}f). \tag{6.13}$$

Hence, in order to conclude, it is enough to show that the above r.h.s. equals

$$\sqrt{2\hat{\alpha}}\coth\left(h\sqrt{2\hat{\alpha}}\right)-\mu\coth(\mu h).$$

By (4.12), we can write the r.h.s. of (6.13) as

$$\lim_{t\downarrow 0}\int_0^\infty n_t(y)\mathbf{E}_y^\mu\left(\mathbb{I}_{T_0<T_h}\left(1-\mathrm{e}^{-\alpha T_0+\alpha t}\right)\right)\mathrm{d}y=\lim_{t\downarrow 0}\left(J_1(t)-\mathrm{e}^{\alpha t}J_2(t)\right),$$

where

$$J_1(t)=\int_0^\infty n_t(y)\mathbf{P}_y^\mu\left(T_0<T_h\right)\mathrm{d}y,$$
$$J_2(t)=\int_0^\infty n_t(y)\mathbf{E}_y^\mu\left(\mathbb{I}_{T_0<T_h}\mathrm{e}^{-\alpha T_0}\right)\mathrm{d}y=\int_0^h n_t(y)\mathbf{E}_y^\mu\left(\mathbb{I}_{T_0<T_h}\mathrm{e}^{-\alpha T_0}\right)\mathrm{d}y.$$

Due to the identities derived at the beginning of the proof of (4.13) we can write

$$\lim_{t\downarrow 0}\int_0^\infty n_t(y)\mathbf{P}_y^\mu\left(T_h<T_0\right)=n\left(U^{h,+}\right)=\frac{\mu\mathrm{e}^{-\mu h}}{\sinh(\mu h)},$$

while due to (6.10) and (6.9), $\int_0^\infty n_t(y)\mathrm{d}y=\frac{2}{\sqrt{2\pi t}}-\mu+o(1)$. Hence, $J_1(t)$ can be written as $2/\sqrt{2\pi t}-\mu\coth(\mu h)+o(1)$. Due to (6.2)

$$\begin{aligned}J_2(t)&=\int_0^h\frac{2y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y+\mu t)^2}{2t}+\mu y}\frac{\sinh\left((h-y)\sqrt{2\hat{\alpha}}\right)}{\sinh\left(h\sqrt{2\hat{\alpha}}\right)}\mathrm{d}y=\frac{\mathrm{e}^{h\sqrt{2\hat{\alpha}}}}{2\sinh\left(h\sqrt{2\hat{\alpha}}\right)}\\&\quad\times\int_0^h\frac{2y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y+\mu t)^2}{2t}-y\left(\sqrt{2\hat{\alpha}}-\mu\right)}\mathrm{d}y\\&\quad-\frac{\mathrm{e}^{-h\sqrt{2\hat{\alpha}}}}{2\sinh\left(h\sqrt{2\hat{\alpha}}\right)}\int_0^h\frac{2y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y+\mu t)^2}{2t}+y\left(\sqrt{2\hat{\alpha}}+\mu\right)}\mathrm{d}y.\end{aligned}$$

Due to (6.10), $J_2(t)$ equals $2/\sqrt{2\pi t}-\sqrt{2\hat{\alpha}}\coth\left(h\sqrt{2\hat{\alpha}}\right)+o(1)$. This allows us to conclude. □

6.4. *Proof of* (4.16)

Due to the Monotone Convergence Theorem

$$\int_U \mathrm{e}^{-\alpha T_h}\mathbb{I}_{T_h<T_0}n(\mathrm{d}f)=\lim_{t\downarrow 0}\int_U \mathrm{e}^{-\alpha T_h}\mathbb{I}_{t<T_h<T_0}n(\mathrm{d}f). \tag{6.14}$$

It is convenient to write the last integral as $A(t)-B(t)$, where

$$A(t)=\mathrm{e}^{-\alpha t}\int_U \mathrm{e}^{-\alpha T_h(\theta_t f)}\mathbb{I}_{T_h(\theta_t f)<T_0(\theta_t f)}\mathbb{I}_{H(f)>t}n(\mathrm{d}f),$$
$$B(t)=\mathrm{e}^{-\alpha t}\int_U \mathrm{e}^{-\alpha T_h(\theta_t f)}\mathbb{I}_{T_h(\theta_t f)<T_0(\theta_t f)}\mathbb{I}_{H(f)>t}\mathbb{I}_{T_h\le t}n(\mathrm{d}f).$$

By (4.12), $\lim_{t\downarrow 0} A(t)$ can be written as $\lim_{t\downarrow 0}(K_1(t)+K_2(t))$, where

$$K_1(t) = \int_0^h n_t(\mathrm{d}y)\mathbf{E}_y^{\mu}\left(\mathrm{e}^{-\alpha T_h}\mathbb{I}_{T_h<T_0}\right),$$
$$K_2(t) = \int_h^\infty n_t(\mathrm{d}y)\mathbf{E}_y^{\mu}\left(\mathrm{e}^{-\alpha T_h}\mathbb{I}_{T_h<T_0}\right) = \int_h^\infty n_t(\mathrm{d}y)\mathbf{E}_{y-h}^{\mu}\left(\mathrm{e}^{-\alpha T_0}\mathbb{I}_{T_0<\infty}\right).$$

Due to (6.3)

$$\begin{aligned} K_1(t) &= \int_0^h \frac{2y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y+\mu t)^2}{2t}+\mu(y-h)}\frac{\sinh\left(y\sqrt{2\hat\alpha}\right)}{\sinh\left(h\sqrt{2\hat\alpha}\right)}\mathrm{d}y \\ &= \frac{\mathrm{e}^{-\mu h}}{2\sinh\left(h\sqrt{2\hat\alpha}\right)}\int_0^h \frac{2y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y+\mu t)^2}{2t}+\mu y}\left(\mathrm{e}^{y\sqrt{2\hat\alpha}}-\mathrm{e}^{-y\sqrt{2\hat\alpha}}\right)\mathrm{d}y. \end{aligned} \tag{6.15}$$

By applying (6.10) to the r.h.s. we get that $K_1(t) = \mathrm{e}^{-\mu h}\sqrt{2\hat\alpha}/\sinh\left(h\sqrt{2\hat\alpha}\right)+o(1)$. By (6.6)

$$\begin{aligned} K_2(t) &= \int_h^\infty \frac{2y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y+\mu t)^2}{2t}+(\mu-\sqrt{2\hat\alpha})(y-h)}\mathrm{d}y \\ &= \mathrm{e}^{\alpha t+h(\sqrt{2\hat\alpha}-\mu)}\int_h^\infty \frac{2y}{\sqrt{2\pi t^3}}\mathrm{e}^{-\frac{(y+\sqrt{2\hat\alpha}t)^2}{2t}}\mathrm{d}y \end{aligned} \tag{6.16}$$

and due to (6.9), $\lim_{t\downarrow 0} K_2(t)=0$. Hence, $A(t)$ converges to $\mathrm{e}^{-\mu h}\sqrt{2\hat\alpha}/\sinh\left(h\sqrt{2\hat\alpha}\right)$ as $t\downarrow 0$. At this point, in order to conclude it is enough to show that $\lim_{t\downarrow 0} B(t)=0$. To this end, we apply the Hölder inequality with exponents $p,q>1$ with $1/p+1/q=1$ and $\widehat{\alpha p}>0$, deriving that

$$B(t) \le \mathrm{e}^{-\alpha t}\left[\int_U \mathrm{e}^{-\alpha p T_h(\theta_t f)}\mathbb{I}_{T_h(\theta_t f)<T_0(\theta_t f)}\mathbb{I}_{H(f)>t}n(\mathrm{d}f)\right]^{1/p} n(T_h\le t)^{1/q}.$$

As $t\downarrow 0$ the first factor in the r.h.s. goes to 1 and the second factor has a finite limit due to our results on $A(t)$ (it is enough to replace α by αp and use that $\widehat{\alpha p}>0$). Hence we only need to prove that $\lim_{t\downarrow 0} n(T_h\le t)=0$. By the Monotone Convergence Theorem it is enough to show that $n(T_h\le 1)<\infty$. However, $n(T_h\le 1)\le n(U^{h,+})$ which is bounded by (4.13). $\square$

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